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Diffraction by a Smooth Object

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Abstract

This paper, on the diffraction of waves by smooth convex obstacles, consists of two parts. In Part I a new geometrical theory of wave propagation is explained and applied to the problem. In Part II the diffraction problem for various obstacles is formulated as a boundary value problem and solved by the usual separation of variables method. The solutions are then asymptotically expanded for large $ka = 2\pi a/\lambda$ (a = obstacle dimension, λ = wavelength) and compared with the results of Part I. In all cases the geometric solution is found to agree with the asymptotic expansion of the exact solution.

The geometrical theory of Part I is developed for solutions of the time reduced Maxwell's equations as well as the reduced wave equation. The diffracting objects are assumed to be of arbitrary convex shape and of any type (e.g., acoustically hard or soft, perfect conductors, dielectrics). In addition to developing and applying the geometrical theory to variously shaped bodies in Part I, a certain feature of the self-consistency of the theory is demonstrated. Some calculations are also carried out to indicate the numerical usefulness of the geometric theory.

The exact solution of the diffraction problem associated with the reduced wave equation or the time reduced Maxwell's equations for the bodies considered in Part I are found and asymptotically expanded in Part II. The results of this section are an extension of previous work in the types of boundary conditions and problems considered. The problem of asymptotically expanding the electromagnetic field diffracted by an imperfectly conducting circular cylinder is treated as well as other vector and scalar diffraction problems associated with spheres and cylinders.

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Part I. Application of the Geometrical Theory of Diffraction

1. Introduction

This paper, on the diffraction of a wave by a smooth convex opaque object of any shape, consists of two parts. In Part I the " geometrical theory of diffraction" [1], a new theory of wave propagation, is explained and applied to the problem. In Part II the diffraction problem for various objects of special shape is formulated as a boundary value problem and solved by the usual separation of variables method. The solutions are then expanded asymptotically for large $ka = \frac{2\pi a}{\lambda}$ (a = obstacle dimension, λ = wavelength) and compared with the results of Part I. In each case the leading terms in the expansion of the solution coincide with the simple geometrical results, thus verifying the geometrical theory.

The " geometrical theory of diffraction" is an extension of geometrical optics which is also based upon the postulate that fields propagate along rays. However, it introduces new kinds of rays, called diffracted rays, in addition to the usual optical ones. In the present case these diffracted rays lie in part on the surface of the diffracting object, so they may also be called surface rays. Furthermore, the new theory assigns a field value, which includes a phase, an amplitude, and, in the electromagnetic case, a direction or polarization to each point on a ray. The total field at a point is postulated to be the sum of the fields on all rays which pass through the point. By using this theory we will obtain an explicit expression for the field produced at any point when a wave hits a smooth convex opaque object.

The theory is developed for both scalar and vector (e.g., electromagnetic) fields and for objects of any type (e.g., acoustically hard or soft, perfect electrical conductors, dielectrics). The results depend upon the nature of the object in an essential way. To simplify the analysis we will ignore rays which go through the object, thereby restricting our results to opaque objects.

An important advantage of the present method is that it does not depend upon separation of variables or any similar procedure. Therefore, the shapes of the objects to which it can be applied are not restricted. But for simplicity we have limited our considerations to smooth convex objects. These include cylinders of convex cross section, which have previously been treated by this same method[2].

From its similarity to geometrical optics, the present method can be expected to yield good results when the wavelength λ is small compared to the obstacle dimensions. As has been pointed out above, this is verified in Part II in certain cases where other descriptions of the field are available for comparison. This method has also been applied to diffraction by an aperture of any shape in a thin screen[3], to diffraction in waveguides[4] and to other problems[5]. In those cases as well as in the present one, it has been found that the results are useful even for wavelengths as large as the relevant dimensions of the scatterer.

Most of our theory is independent of the nature of the field under consideration. Therefore, the consequences of this part of the theory are the same for all fields to which the theory applies. For example, the procedure for locating the rays is the same for all fields. On the other hand, some aspects of the theory depend specifically on the kind of field and the type of obstacle. Consequently these aspects must be considered separately in each case. In particular, the decay rates associated with the field on a surface ray, the diffraction coefficients, and the polarization effects are in this category. Therefore, it will be convenient first to present the theory for a scalar field and then to modify the analysis where necessary, in order to treat vector fields.

The problem which we consider is that of finding the time harmonic field which results when a given incident time harmonic field impinges upon a smooth object in a homogeneous medium. The incident field will be characterized in terms of the incident rays and the field on these rays.

2. Ray tracing

According to geometrical optics, the only rays which occur in the present problem are the incident rays, the reflected and refracted rays which they produce on hitting the object, and the resulting multiply-reflected and refracted rays. All of these rays are straight lines and their directions are determined by the laws of reflection and refraction. We will ignore the refracted rays since we assume the body to be opaque. Then, since the body is convex, no multiply-reflected rays can occur.

In the geometrical theory of diffraction, additional rays, called diffracted rays, are introduced. In the present problem these rays are produced by incident rays which are tangent to the surface of the body [see Figure 1]. Each tangent ray "splits" at the point of tangency. One part continues along the path of the incident ray. Another part travels along the surface of the body as a surface ray. This surface ray is a geodesic or shortest path on the body surface. At each point it splits again, one part continuing along the surface ray and the other part leaving the surface along the tangent to the surface ray. Thus a single grazing incident ray gives rise to infinitely many diffracted rays, one of which is shed at each point of the surface ray.

Diffracted rays can be characterized by an extended form of Fermat's Principle. In the present case the relevant extension is this: The diffracted rays connecting two points Q and P in the exterior of the body are those curves joining Q and P which have stationary (optical) length among all curves from Q to P having an arc on the body surface. From this principle it follows that a diffracted ray consists of a straight line segment from Q to the body surface, a geodesic arc along the surface and a straight line segment from the body to P. It also follows that the straight line segments are tangent to the arc. These are just the properties of diffracted rays which were described above.

Diffracted rays penetrate the usual shadow regions of geometrical optics and account for the non-zero fields there. They also modify the field in geometrically lit regions.

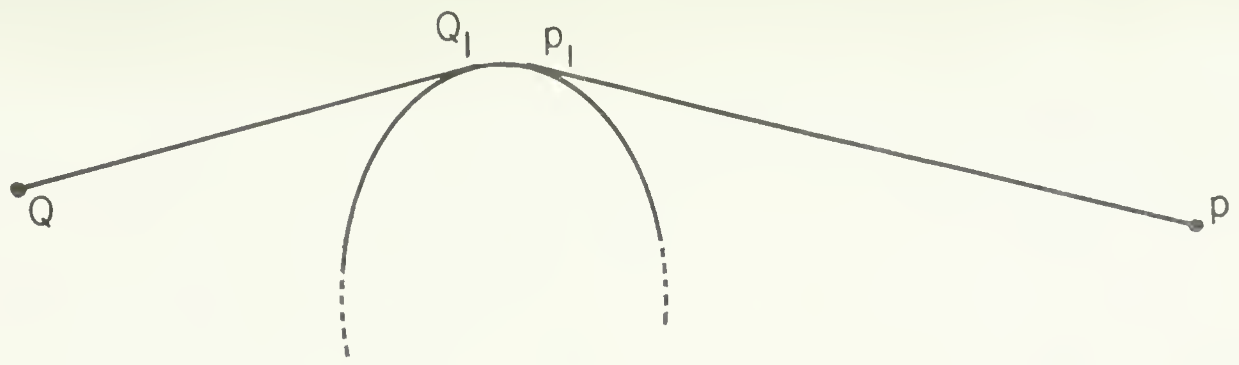

Figure 1: A single diffracted ray from Q to P. The ray consists of a straight line tangent to the surface at Q_1, a geodesic arc on the surface from Q_1 to P_1, and a straight diffracted ray from P_1 to P.

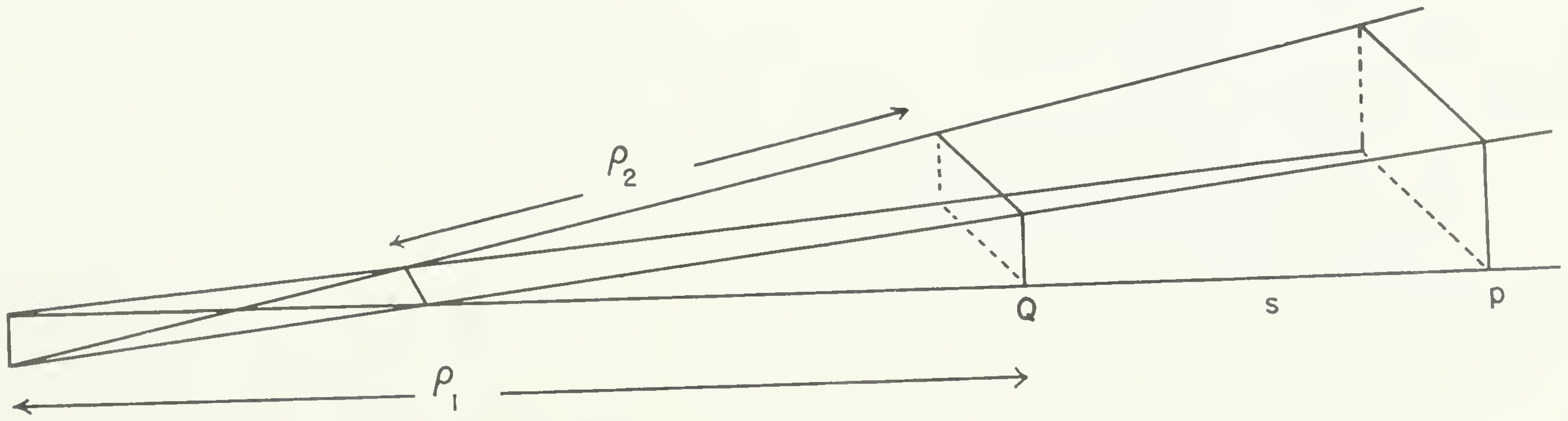

Figure 2: The principal radii of curvature of the wavefront through a point Q. Since the wavefronts are parallel surfaces the principal radii of curvature through P are $\rho_1 + s$ and $\rho_2 + s$.

3. Field calculation

In the geometrical theory of diffraction, just as in ordinary geometrical optics, a field value is associated with each ray. This field value is composed of a phase function $\phi(s)$ and an amplitude $A(s)$, both of which may vary with distance s along the ray. For vector fields there is also an orientation or polarization which we will treat later by permitting $A(s)$ to be a vector. The phase ϕ is determined by the optical law $d\phi/ds = 1$ from which it follows that $\phi(s) = \phi_o + s$. Here ϕ_o is the phase at the point Q from which s is measured; moreover, s increases in the direction of propagation.

The amplitude $A(s)$ is determined by the optical form of the principle of conservation of energy. This principle states that in time harmonic fields the energy flux is the same through every cross section of a tube of rays. For a narrow tube, having cross-sectional area $d\sigma$ at a point P, this flux is proportional to $A^2 d\sigma$. Here A is the amplitude of the field at the point P of a ray contained in the tube. If A_o and $d\sigma_o$ are the amplitude and cross-sectional area at another point Q of the same ray, then we have $A^2 d\sigma = A_o^2 d\sigma_o$. Thus $A = A_o(d\sigma_o/d\sigma)^{1/2}$. To evaluate the area ratio we observe that $d\sigma$ is just the area cut out by the tube on that wavefront ϕ = constant which passes through P. Similarly, $d\sigma_o$ is the area cut out on the wavefront $\phi = \phi_o$ through Q. If s denotes the distance from P to Q, a simple computation shows that [see Figure 2] since the rays are straight lines,

$$\frac{d\sigma_o}{d\sigma} = \frac{\rho_1\rho_2}{(\rho_1+s)(\rho_2+s)} . \tag{1}$$

Here ρ_1 and ρ_2 are the principal radii of curvature of the wavefront through Q.

Upon combining the above results for ϕ and A, we find that the field $u(P)$ associated with the straight ray from Q to P is

$$u(P) = A_o\left[\frac{\rho_1\rho_2}{(\rho_1+s)(\rho_2+s)}\right]^{1/2} e^{ik(\phi_o+s)} . \tag{2}$$

Here $k = \omega/c$ is the propagation constant, ω is the angular frequency of the field, and c is its propagation speed. We see that the field at P is determined in terms of the field $u(Q) = A_0 e^{ik\phi_0}$ at Q. The incident field $u_i(P)$ is of the form (2) since the incident rays are straight lines.

To determine the field on a reflected ray we assume that at the point of reflection this field is proportional to the incident field. The coefficient of proportionality is the reflection coefficient R which depends upon the nature of the field, the properties of the reflector at the point of reflection, the angle of incidence and, in some cases, the wavelength $\lambda = 2\pi/k$. For vector fields R is a matrix. Because R depends only upon properties at the point of reflection, it can be determined from the analysis of the simpler problem of reflection of a plane wave from an infinite plane interface. Since R is known for many types of fields and reflectors, we will not consider it further. The calculation of the radii of curvature of the reflected wavefront from the knowledge of the incident wavefront and the reflector shape has also been performed before[6]. Therefore, we may assume that the reflected field $u_r(P)$ can be calculated completely. It will also be of the form (2) since the reflected rays are also straight lines.

We denote by $u_g(P)$ the geometrical optics field at P. This is the sum of the fields - each of the form (2) - on the various incident and reflected rays through P. In the shadow $u_g = 0$ since there are no incident or reflected rays there.

Let us now apply the preceding considerations to a diffracted ray such as that shown in Figure 1. Such a ray consists of a straight incident ray through Q tangent to the surface at Q_1, a geodesic arc on the surface from Q_1 to P_1, and a straight diffracted ray from P_1 to P. The field on the incident ray up to the point Q_1 is given by equation (2) above. We now assume that $u_d(Q_1) = A_d(Q_1)e^{ik\phi_d(Q_1)}$, the field on the diffracted (or surface) ray at Q_1, is proportional to $u_i(Q_1) = A_i(Q_1)e^{ik\phi_i(Q_1)}$, the field on the incident ray at Q_1*. We also assume that the phases are equal so $\phi_d(Q_1) = \phi_i(Q_1)$. Thus the proportionality becomes

$$A_d(Q_1) = D(Q_1)A_i(Q_1) \tag{3}$$

We call the coefficient of proportionality $D(Q_1)$ the diffraction coefficient. We assume that it depends upon the nature of the field, the properties of the object at Q_1 and the propagation constant k. In the case of vector fields $D(Q_1)$ is a matrix.

Before attempting to determine the diffraction coefficient $D(Q_1)$, let us consider how the diffracted field varies along the diffracted (or surface) ray. If t denotes distance along the diffracted ray from Q_1 then the previous considerations yield, for the phase $\phi_d(t)$, the result $\phi_d(t) = \phi_i(Q_1) + t$. To determine the amplitude $A_d(t)$ we apply the energy conservation principle to a narrow strip of diffracted rays on the surface of the object (see Figure 3). Let us denote the width of the strip at distance t from Q_1 by $d\sigma(t)$. Then the energy flux through a cross section of the strip is proportional to $A_d^2(t)\, d\sigma(t)$. At the slightly greater distance $t + dt$ the flux will be smaller because diffracted rays are shed in the interval from t to $t + dt$.

The shedding of these rays corresponds to radiation of energy from the strip of surface rays. We assume that the radiated energy is proportional to $A_d^2(t)$ and to the area $d\sigma(t)dt$ from which radiation occurs. Thus the energy conservation principle becomes

$$A_d^2(t+dt)d\sigma(t+dt) - A_d^2(t)d\sigma(t) = -2\alpha(t)\, A_d^2(t)d\sigma(t)dt. \tag{4}$$

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[*] The case in which the field, or some component of it, is required by a boundary condition to be zero on the surface will be considered separately later. The results in that case are the same as those described here, but u_d then has a different interpretation.

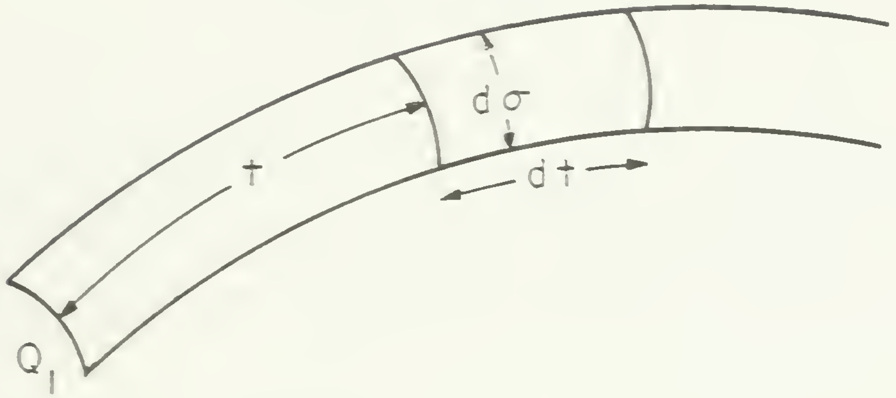

Figure 3: A band of diffracted rays on the surface of a diffracting object. The geodesic distance from the point Q_1 is t, while $d\sigma$ is the width of the band.

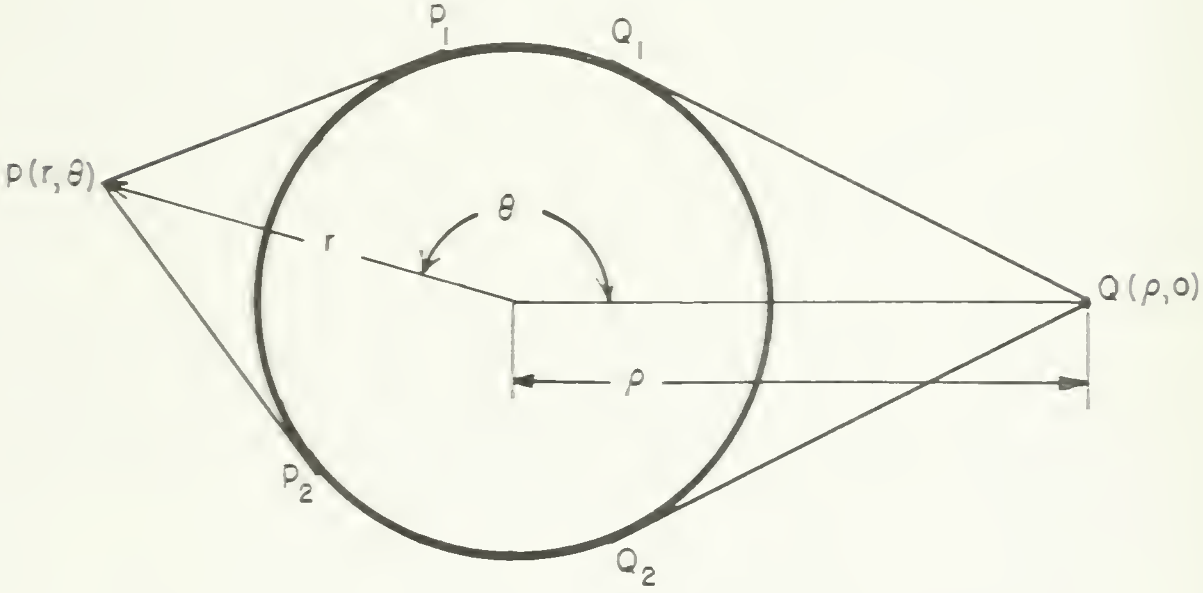

Figure 4: The two families of rays emanating from a line source located at Q and diffracted around a circular cylinder to a point P. The upper family is incident on the cylinder at Q_1 and leaves the cylinder tangentially at P_1, while the lower family is incident on the cylinder at Q_2 and leaves the cylinder tangentially at P_2.

Here $2\alpha(t)$ is a proportionality factor relating the change of flux to the energy radiated. The physical nature of the radiation process suggests that $\alpha(t)$ depends only upon local properties of the surface and the field. We will consider the determination of $\alpha(t)$ later. Now we divide (4) by dt and let dt tend to zero, obtaining

$$(5)\qquad \frac{d}{dt}\left[A_d^2 d\sigma\right] = -2\alpha\left[A_d^2\, d\sigma\right] .$$

Upon integrating (5) we find

$$(6)\qquad A_d(t) = A_d(0)\left(\frac{d\sigma_0}{d\sigma}\right)^{1/2} \exp\left[-\int_0^t \alpha(t)dt\right] .$$

Here $d\sigma_0/d\sigma$ is the ratio of the strip width at $t = 0$ to that at t, or rather the limit of this ratio as the strip width tends to zero. The point $t = 0$ is the point of tangency Q_1 so $A_d(0) \equiv A_d(Q_1)$ is given by (3) and $d\sigma_0$ may be called $d\sigma(Q_1)$. If we insert this value into (6), replace $d\sigma_0$ by $d\sigma(Q_1)$, and include the phase, we obtain for the diffracted field on a surface ray

$$(7)\qquad u_d(t) = D(Q_1)A_i(Q_1)\left[\frac{d\sigma(Q_1)}{d\sigma}\right]^{\frac{1}{2}} \exp\left\{ ik\left[\phi_i(Q_1) + t\right] - \int_0^t \alpha(t)dt \right\} .$$

It is to be noted that $u_d(t)$ is proportional to the incident field at Q_1.

To obtain the diffracted field at a point P off the surface we first apply (7). This yields the diffracted field at the point of tangency P_1 from which a tangent ray travels to P. Then we make use of (2), which applies to a straight ray, to describe the variation of the field from P_1 to P. However, we cannot apply (2) directly since one of the radii of the diffracted wavefront - say ρ_2 - is zero at P_1. This is so because the surface of the object is a caustic of the family of diffracted rays. Therefore we proceed as follows:

Let Q denote a point on the diffracted ray from P_1 to P. Then (2) expresses the field at P in terms of that at Q. Now let Q tend to P_1. Then ρ_2 tends to zero, ρ_1 tends to some finite limit, ϕ_o tends to $\phi_d(P_1)$ and s tends to the distance from P_1 to P while u(P) remains constant. It follows from (2) that A_o must become infinite in such a way that $A_o\ \rho_2^{1/2}$ has a finite limit. If we denote this limit by $A_d'(P_1)$ then (2) becomes

$$(8)\qquad u_d(P) = A_d'(P_1)\left[\frac{\rho_1}{s(\rho_1+s)}\right]^{1/2}\exp\left\{ik\left[\phi_d(P_1)+s\right]\right\}.$$

The expression (8) for $u_d(P)$ becomes infinite at the surface of the object, at which $s = 0$ - a consequence of the fact that this surface is a caustic of the diffracted rays. This is an instance of the failure of the present theory to yield the correct result for the field at or near a caustic. As a consequence the result must be appropriately modified in such places, and we will show how later. At present, however, this fact prevents us from identifying the field given by (8) with that given by (7), when both are evaluated at P_1. If such an identification were possible we would be able to determine the amplitude factor $A_d'(P_1)$ in (8). Instead we will determine this factor by observing that it must be proportional to the amplitude in (7) evaluated at P_1. We write this proportionality as

$$(9)\qquad A_d'(P_1) = D(P_1)D(Q_1)A_i(Q_1)\left[\frac{d\sigma(Q_1)}{d\sigma(P_1)}\right]^{\frac{1}{2}}\exp\left[-\int_0^t \alpha(\tau)d\tau\right].$$

We call the proportionality factor $D(P_1)$ a diffraction coefficient. It depends upon the nature of the field and the object in the neighborhood of P_1. We further assume that $D(P_1)$ is the same function of these variables as is $D(Q_1)$. This assumption is based upon the reciprocity principle - that a source at Q produces the same field at P as a source at P produces at Q. With our assumption this principle is satisfied, but not otherwise. Upon inserting (9) into (8) we obtain for the diffracted field at P the result

$$(10)\qquad u_d(P) = A_i(Q_1)D(P_1)D(Q_1)\left[\frac{d\sigma(Q_1)}{d\sigma(P_1)}\right]^{\frac{1}{2}}\left[\frac{\rho_1}{s(\rho_1+s)}\right]^{\frac{1}{2}}\exp\left\{ik\left[\phi_i(Q_1)+t+s\right]-\int_0^t \alpha(t)dt\right\}.$$

In the derivation of this result we excluded cases in which the field or some component of it was required, by a boundary condition, to vanish on the surface. This was necessary because our analysis dealt with the function $A_d(t)$, the diffracted field amplitude on the surface. However, as we have seen, the surface is a caustic of the diffracted field. From this it follows that the field is much stronger in a surface layer some few wavelengths thick than it is at points a large number of wavelengths from the surface. Therefore, the preceding argument will still apply and include the previously excluded cases if $A_d(t)$ is any measure of the field amplitude in this layer. The field within the layer will be characterized by a definite profile along a surface normal, i.e., a definite variation with distance from the surface. The amplitude of the field at any point other than a zero of the profile can serve as the measure of the field amplitude in this layer.

In general the field within the caustic layer is composed of a number of different modes, each with its own profile. Each mode is characterized by its own amplitude, A_{dm} and therefore its own diffraction coefficient $D_m(Q_1)$ and proportionality factor for radiation $\alpha_m(t)$. Consequently the diffracted field (10), which represents the contribution of only one mode, must be replaced by

$$(11)\quad u_d(P) = A_i(Q_1)\exp\left\{ik\left[\phi_i(Q_1)+t+s\right]\right\}\left[\frac{d\sigma(Q_1)}{d\sigma(P_1)}\right]^{\frac{1}{2}}\left[\frac{\rho_1}{s(\rho_1+s)}\right]^{\frac{1}{2}}$$

$$\sum_m D_m(P_1)D_m(Q_1)\exp\left[-\int_0^t \alpha_m(\tau)\,d\tau\right] \quad .$$

Equation (11) yields the diffracted field associated with a single diffracted ray. If more than one ray passes through P, then $u_d(P)$ is a sum of terms of the above form, one term for each diffracted ray through P. The total field at P is then

$$(12)\qquad u(P) = u_g(P) + u_d(P).$$

The above result enables us to calculate the field at any point which does not lie on a caustic or shadow boundary. This calculation involves the incident field, various geometrical quantities, the reflection coefficient R, the diffraction coefficients $D_m(Q_1)$ and the decay exponents α_m. Our result is not yet complete because we have not shown how to determine D_m and α_m. This is done in the next section. We will later show how to calculate the field on or near the surface, which is a caustic, and at other caustics.

4. Diffraction by a circular cylinder

From (3), which defines the diffraction coefficient $D_m(Q_1)$, it follows that this coefficient is dimensionless. Therefore, if it depends upon the propagation constant k, it must be a function of it in some dimensionless combination ka where a has the dimension of length. Let us choose a to be the radius of curvature of the object at Q_1 in the plane of incidence, i.e., in the plane containing the incident ray and the normal to the surface. We now assume that the main or leading term in $D_m(Q_1)$ does not depend upon any other geometrical property of the object. In particular it does not depend upon the radius of curvature of the surface in the plane normal to the plane of incidence. This assumption appears to be in accord with the ray nature of the field propagation. Of course small additional terms may involve other geometrical properties.

The decay exponent α_m has the dimension of $(\text{length})^{-1}$. We will assume that it depends upon k and upon a, the radius of curvature of the surface in the normal plane tangent to the ray. This radius is the reciprocal of the geodesic curvature of the ray. We will see in a number of examples that the most important (i.e., largest) term in the expression for α_m involves no other geometrical property of the surface. However, small correction terms do involve other geometrical properties.

On the basis of the preceding considerations it follows that the main terms in D_m and α_m can be determined from the field diffracted by any object of simple shape. For this purpose one may consider diffraction of the field due to a line source by a circular cylinder of radius a. If the line source is parallel to the axis of the cylinder this problem simplifies to a two-dimensional one. When this field is compared with that given by (12) the coefficients D_m and α_m can be found as functions of a.

In order to effect this comparison it is first necessary to work out explicitly the field given by (12). This will be done in the remainder of this section. Then it is necessary to obtain another expression for the field by formulating and solving an appropriate boundary value problem and expanding the solution asymptotically for large values of ka. This is done in Part II, Section 2, for fields satisfying the reduced wave equation and four different boundary conditions. At the end of the present section the two expressions are compared and a table of values of D_m and α_m for the four cases is obtained.

Consider a circular cylinder with its generators parallel to the z axis. Let an (r,θ) coordinate system be located in the (x,y) plane with $\theta = 0$, $r = \rho$ corresponding to the location of a line source (see Figure 4). The wavefronts associated with the line source are concentric cylinders, while the rays are the straight lines emanating from the source and orthogonal to the wavefronts. By using these rays and wavefronts in (2) we obtain an expression for the incident field produced by the source. If R is the perpendicular distance from any point to the source, then this incident field at the point is*

$$u_i = \frac{i}{4} \left(\frac{2}{k\pi R}\right)^{1/2} e^{ikR - i\pi/4} \quad . \tag{13}$$

Due to the symmetry of the problem, it is clear that the geodesics are arcs of the generating circle of the cylinder. Furthermore, the quantity $d\sigma(Q_1)/d\sigma$ has the value unity since a band of diffracted rays does not spread on the cylinder. Also the radius of curvature ρ_1 in (11) is infinite so that $\left[\rho_1/(\rho_1+s)s\right]^{1/2}$ assumes its limiting value $s^{-1/2}$. Moreover, because the geodesic curvature of a ray is constant, the quantities $\alpha_m(t)$ are constants and $D_m(Q_1) = D_m(P_1)$. We now introduce all of these simplifications into (11) and use (13) for $u_i(Q_1)$ noting that R at Q_1 is $(\rho^2-a^2)^{1/2}$. Then (11) becomes

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* The constant in (13) has been adjusted so that u_i represents the field of a source of unit strength, i.e., one for which the exact field is $\frac{i}{4} H_o^{(1)}(kR)$.

$$(14) \qquad u_d(r,\theta) = \left[8\pi k s(\rho^2-a^2)^{1/2}\right]^{-1/2} \exp\left[ik\left[(\rho^2-a^2)^{1/2}+s\right]+\frac{i\pi}{4}\right] \sum_m D_m^2 \, e^{(ik-\alpha_m)t} .$$

Equation (14) gives the field associated with any ray from Q to P having an arc of length t on the cylinder. For the ray QQ_1P_1P, the length t has the value $t_o = a\theta - a\cos^{-1}(a/\rho) - a\cos^{-1}(a/r)$, while $s = (r^2-a^2)^{1/2}$. In addition to this ray, all of the rays which are tangent to the cylinder at Q_1 and encircle the cylinder n times before emerging at P_1 are also diffracted rays through P. For these rays t has the value $t_n = t_o + 2n\pi a$. Inserting this value of t_n into (14) and summing over n, we find that the diffracted field at P due to those rays which are tangent at Q_1 is given by

$$(15) \qquad u_d(r,\theta) = (8\pi k)^{-1/2}(r^2-a^2)^{-1/4}(\rho^2-a^2)^{-1/4}\exp\left[ik\left(\left[\rho^2-a^2\right]^{1/2}+\left[r^2-a^2\right]^{1/2}\right)+\frac{i\pi}{4}\right] \sum_m D_m^2 \, e^{(ik-\alpha_m)t_o}\left[1-\exp\left[2\pi(ika-a\alpha_m)\right]\right]^{-1} .$$

Along with the family of rays described above we also have rays which are tangent to the cylinder at Q_2, encircle the cylinder n times and then emerge tangentially at P_2 and pass through P. Computing the field along these rays yields a result of the form (15) with θ replaced by $2\pi - \theta$. Adding this result to (15), we obtain for the total diffracted field at P

$$u_d(r,\theta) = (8\pi k)^{-1/2}(r^2-a^2)^{-1/4}(\rho^2-a^2)^{-1/4}\exp\left[ik\left(\left[\rho^2-a^2\right]^{1/2}+\left[r^2-a^2\right]^{1/2}\right)+\frac{i\pi}{4}\right]$$

$$(16) \qquad \sum_m D_m^2 \; \frac{\exp\left[(ika-a\alpha_m)\theta\right] + \exp\left[(ika-a\alpha_m)(2\pi-\theta)\right]}{1-\exp\left[2\pi(ika-a\alpha_m)\right]}$$

$$\cdot \exp\left[-(ika-a\alpha_m)\left(\cos^{-1}\left(\frac{a}{r}\right)+\cos^{-1}\left(\frac{a}{\rho}\right)\right)\right] .$$

In Part II, Section 2 we have obtained the exact solution of the boundary value problem for diffraction of a scalar field by a circular cylinder. The field satisfies the reduced wave equation and satisfies one of four different conditions at the cylinder surface, corresponding to different physical situations. We have also expanded these solutions asymptotically for ka large. In all four cases the leading terms in the expansion of the diffracted field have exactly the form given by (16). From this comparison we obtain the following values for α_m and D_m in these cases

$$(17) \qquad \alpha_m = ik - i\nu_m \, a^{-1}$$

$$(18) \qquad D_m = e^{5\pi i/8} \left(\frac{2\pi}{k}\right)^{1/4} \left[\frac{\Omega H^{(2)}_{\nu_m}(ka)}{\frac{\partial}{\partial\nu}\,\Omega H^{(1)}_{\nu_m}(ka)}\right]^{1/2} .$$

The expressions on the right sides of these equations are defined in Part II, Section 2. Values of α_m and D_m, based on these equations, are given in Table I for the four types of boundary conditions which are also listed in the table.

5. Electromagnetic diffraction

In the electromagnetic case the field is represented by two vector functions, the electric and magnetic fields. Therefore, as we pointed out above, each of these fields is characterized by a phase $\phi(s)$ and a vector amplitude $A(s)$. We

Case	Boundary Conditions	$\alpha_m = e^{-\frac{i\pi}{6}}\left(\frac{k}{6a^2}\right)^{1/3} q_m$	$e^{-\frac{5\pi i}{4}}\left(\frac{k}{2\pi}\right)^{1/2} D_m^2 = \frac{\Omega H_{\nu_m}^{(2)}(ka)}{\frac{\partial}{\partial\nu}\Omega H_{\nu_m}^{(1)}(ka)}$
I	$u = 0$	$A(q_m) = 0$	$\frac{\pi e^{\frac{5\pi i}{6}}}{6}\left(\frac{ka}{6}\right)^{1/3} \frac{1}{[A'(q_m)]^2}$
II	$\frac{\partial u}{\partial n} = 0$	$A'(q_m) = 0$	$\frac{\pi e^{\frac{5\pi i}{6}}}{2}\left(\frac{ka}{6}\right)^{1/3} \frac{1}{q_m A^2(q_m)}$
III	$\frac{\partial u}{\partial n} = -ik\ Zu$	$\frac{A'(q_m)}{A(q_m)} = e^{\frac{5\pi i}{6}}\left(\frac{ka}{6}\right)^{1/3} Z$	$\frac{\pi e^{\frac{5\pi i}{6}}}{6}\left(\frac{ka}{6}\right)^{1/3}\left[\left\{A'(q_m)\right\}^2 + q_m A^2(q_m)/3\right]^{-1}$
IV	$\frac{\partial u}{\partial n}$ continuous $[bu]_{r=a_+} = [b_1 u]_{r=a_-}$	$\frac{A'(q_m)}{A(q_m)} = e^{\frac{5\pi i}{6}}\left(\frac{ka}{6}\right)^{\frac{1}{3}}\left[\left(\frac{k_1}{k}\right)^2 - 1\right]^{\frac{1}{2}} b/b_1$	$\frac{\pi e^{\frac{5\pi i}{6}}}{6}\left(\frac{ka}{6}\right)^{1/3}\left[\left\{A'(q_m)\right\}^2 + q_m A^2(q_m)/3\right]^{-1}$

TABLE I

assume that the function ϕ is the same for both fields and is given by the method previously described. We also assume that the direction of each amplitude vector remains constant along a straight ray and that its magnitude is given by the previous construction. Therefore, on a straight ray each field is given by (2) with A_o a vector. Finally we assume that the field vectors are orthogonal to each other and to the ray.

In order to describe the field on a diffracted ray, we first introduce $\vec{d}_1$, the unit tangent to the ray, $\vec{d}_2$ the outward unit normal to the surface and $\vec{d}_3 = \vec{d}_1 \times \vec{d}_2$. Since the diffracted ray is a geodesic, $\vec{d}_2$ lies along the principal normal to the ray and $\vec{d}_3$ lies along the binormal. At a point P on the straight diffracted ray which leaves the surface at P_1 we define the triad of vectors by translating those at P_1 to P. We denote the components of the electric field in the $\vec{d}_1$, $\vec{d}_2$ and $\vec{d}_3$ directions by E_1, E_2 and E_3 and use a similar notation for the magnetic field H.

We now assume that the components E_2 and E_3 propagate independently of each other and obey the principles satisfied by scalar fields, while E_1 is zero. Therefore the diffracted fields E_2 and E_3 are each given by (11) with $A_o \exp\left[ik\phi_i(Q_1)\right]$ replaced by $E_2(Q_1)$ or $E_3(Q_1)$. However, the diffraction coefficients D_m and the decay exponents α_m are different for the two components.

On the basis of the above assumptions it is possible to express the diffracted electromagnetic field in terms of two scalar fields. First we determine all the diffracted rays which pass through a given point P. Each of these rays originates at a point Q_n on the surface and leaves the surface at some point P_n. Then we define $w(P,Q_n)$ and $v(P,Q_n)$ as the values at P of two scalar fields due to all diffracted rays which originate at Q_n and leave the surface at P_n. The corresponding incident fields must have zero phase and unit amplitude at Q_n. The field v is constructed by using the values of D_m and α_m appropriate to the normal component E_2 while w is constructed with the values of D_m and α_m appropriate to the tangential component E_3.

Now we can write the diffracted field as

$$E^d(P) = \sum_n \left[E_2^i(Q_n)\, v(P,Q_n)\, \vec{d}_2(P_n) + E_3^i(Q_n)\, w(P,Q_n)\, \vec{d}_3(P_n) \right]. \tag{19}$$

In (19) E_2^i and E_3^i denote the components of the incident field at Q_n. Equation (19) is just an analytic statement of the assumptions described above.

For a perfectly conducting object we assume that D_m and α_m are the same for the tangential component E_3 as for a scalar field satisfying the condition $u = 0$ on the surface. These values are given in Table I, Case I. Similarly, we assume that for the normal component F_2 they are the same as for a scalar field satisfying the condition $\partial u/\partial n = 0$ on the surface. These values are also given in Table I, Case II.

For an imperfectly conducting object we assume that the D_m and α_m depend upon the torsion as well as the radius of curvature of the diffracted ray. If we denote the torsion of the diffracted ray at the surface of the body by τ then the values of D_m and α_m for the tangential component of the field are given in Table I, Case IV with

$$b/b_1 = \mu\,\mu_1^{-1}\, f(k,k_1,a\tau).$$

Similarly, the values of α_m and D_m for the normal component of the field are given in Table I, Case IV with

$$b/b_1 = k^2 \mu_1 k_1^{-2} \mu^{-1} f(k,k_1,a\tau).$$

In the above the function f is given by

$$f(k,k_1,a\tau) = k_1^2 \left\{ (k_1^2 - k^2)(a\tau)^2 + k_1^2 \right\}^{-1}.$$

Notice that for $\tau = 0$, i.e., for plane diffracted rays, the function f is identically equal to one. Also, if we use the above values for b/b_1 in Table I and then let k_1 tend to infinity, we see that we obtain our previous results for perfectly conducting objects.

6. Scalar diffraction by a sphere

Let us consider the diffraction of a scalar spherical wave by a sphere of radius a. Let the source of the wave be at $r = \rho > a$, $\theta = 0$ in a polar coordinate system with the origin at the center of the sphere (see Figure 5). The incident field u_i at a distance R from the source is found from (2) to be $e^{ikR}/4\pi R$. The constant has been chosen to correspond to a source of unit strength. At the point of tangency of an incident ray on the sphere, $R = (\rho^2-a^2)^{1/2}$.

To apply (11) we observe that, just as in the case of the cylinder, the function D_m and α_m are constants. The radius a entering into the determination of these quantities is simply the radius a of the sphere. The ratio $d\sigma(Q_1)/d\sigma(P_1)$ reduces to the ratio a_o/a_1 of the radii of the latitude circles at points Q_1 and P_1 respectively. In order to facilitate the calculation of this quantity we have drawn in Figure 5 a cross section of the sphere through the plane containing the diffracted rays. A simple calculation then shows that

$$a_o = a(\rho^2-a^2)^{1/2}\ \rho^{-1} \tag{20}$$

$$a_1 = ar^{-1}(a \sin\theta - s\cos\theta)\ . \tag{21}$$

From Figure 5 it is also apparent that the distance t which is the length of the arc of the circle from Q_1 to P_1 is given by

$$t = a\theta - a\cos^{-1}(a/r) - a\cos^{-1}(a/\rho)\ . \tag{22}$$

In Appendix I it is shown that ρ_1, the non-zero principal radius of curvature of the diffracted wavefront on the body, is

$$\rho_1 = a(a\sin\theta - s\cos\theta)(a\cos\theta + s\sin\theta)^{-1}. \tag{23}$$

From (20), (21), and (23) we obtain

$$\left[\frac{d\sigma(Q_1)}{d\sigma(P_1)}\right]^{\frac{1}{2}} \left[\frac{\rho_1}{s(\rho_1+s)}\right]^{\frac{1}{2}} = \left[\frac{a}{\rho r \sin\theta}\right]^{\frac{1}{2}} \left[\frac{\rho^2-a^2}{r^2-a^2}\right]^{\frac{1}{4}}\ . \tag{24}$$

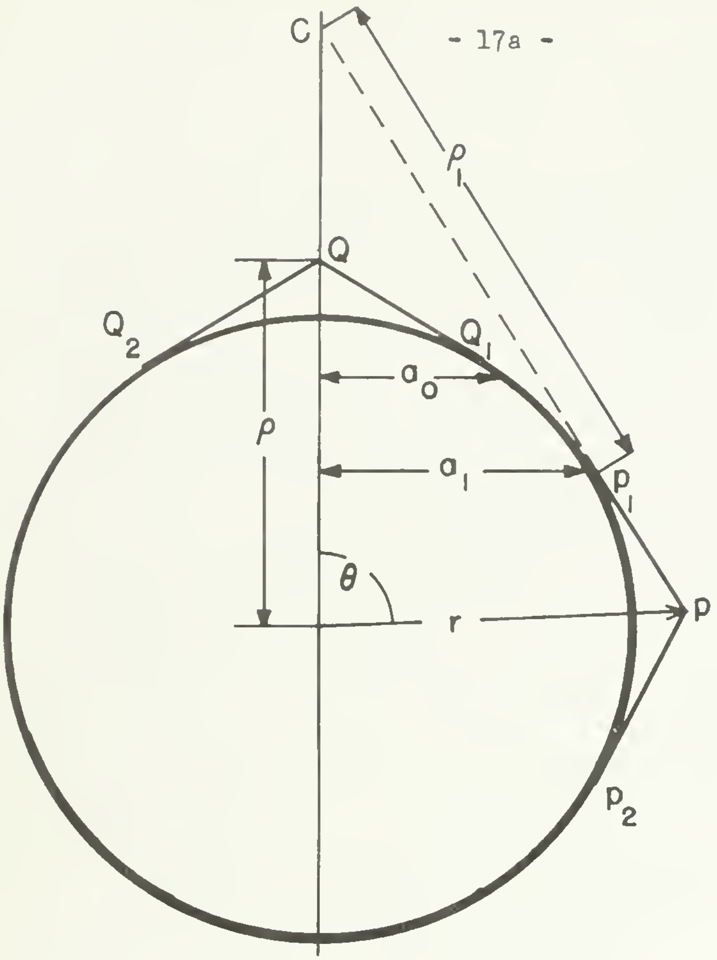

Figure 5: The two families of rays emanating from a point source located at Q and diffracted around a sphere to a point P in the dark region. The cross-section of the sphere shown is the plane of the polar axis and the line joining Q and P. One family of rays is incident at Q_1 and leaves at P_1, while the other family is incident at Q_2 and leaves at P_2.

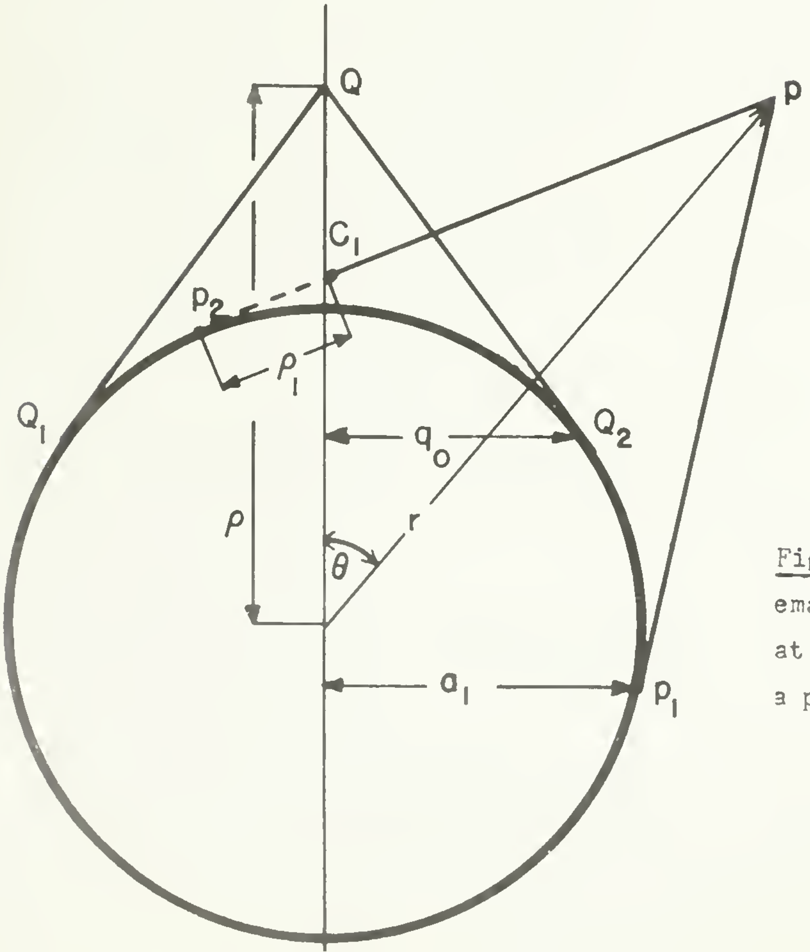

Figure 6: The two families of rays emanating from a point source located at Q and diffracted around a sphere to a point P in the illuminated region.

Upon substituting (22) and (24) together with the value of the incident field on the body into (11) we obtain

$$(25)\qquad u = \frac{a^{1/2}\exp\left[ik\left\{(\rho^2-a^2)^{1/2}+(r^2-a^2)^{1/2}\right\}\right]}{4\pi(\rho r|\sin\theta|)^{1/2}(r^2-a^2)^{1/4}(\rho^2-a^2)^{1/4}}\sum_m D_m^2$$

$$\exp\left[(ik-\alpha_m)a\left\{\theta - \cos^{-1}(a/r) - \cos^{-1}(a/\rho)\right\}\right] .$$

To obtain the complete expression for the diffracted field at P, we must include diffracted rays which are tangent at Q_1 and encircle the sphere n times before leaving at P_1. Then we must also include a similar family of rays tangent at Q_2 leaving the sphere at P_2. As in the case of the cylinder, the field at P due to all the rays tangent at Q_1 can be obtained by summing a geometrical series. In the case of a sphere, however, a phenomenon which is not present in the case of the cylinder appears. All of the diffracted rays, except the ones sketched, pass through foci at the north and south poles. We postulate that the phase decreases by $\pi/2$ as a ray passes through a focus, in accordance with optical experience. The series which must be summed is then $\exp\left[2n\pi a(ik-\alpha_m)-in\pi\right]$ from $n = 0$ to $n = \infty$. Upon summing this series and using the result in (25), we obtain for the contribution to the field at P from all these rays which are tangent at Q_1 and encircle the sphere any number of times

$$(26)\qquad w = \frac{a^{1/2}\exp\left[ik\left\{(\rho^2-a^2)^{1/2}+(r^2-a^2)^{1/2}\right\}\right]}{4\pi(\rho r|\sin\theta|)^{1/2}(r^2-a^2)^{1/4}(\rho^2-a^2)^{1/4}}\sum_m D_m^2\left\{1+\exp\left[2\pi a(ik-\alpha_m)\right]\right\}^{-1}$$

$$\exp\left[(ik-\alpha_m)a\left\{\theta - \cos^{-1}(a/r) - \cos^{-1}(a/\rho)\right\}\right] .$$

The field due to rays tangent at Q_2 is obtained from (26) by replacing θ by $2\pi-\theta$ and multiplying the resulting expression by $-i$. Thus the total diffracted field is

(27) $$u_d(r,\theta) = w(r,\theta) - i\, w(r,2\pi-\theta).$$

When use is made of (17) and (18), it is easily seen that (27) agrees exactly with (10) of Part II which was obtained by asymptotically expanding the solution of the corresponding boundary value problem.

We have tacitly assumed (see Figure 5), in constructing the diffracted field as given by (27), that the point of observation is in the geometrically dark region. As a further check on the geometrical construction of the diffracted field we construct the diffracted field in the geometrically lit region. The geometry is shown in Figure 6.

Simple geometric calculations show that in this case the distance t_o from Q_1 to P_1 along the circle has the value

$$t_o = (2\pi-\theta)\, a - a \cos^{-1}(a/r) - a \cos^{-1}(a/\rho).$$

The value of a_o is again given by (20), while

(28) $$a_1 = ar^{-1}(a \sin\theta + s\cos\theta).$$

In Appendix I it is shown that ρ_1 is equal to the straight line distance between P_1 and C_1. From Figure 6 this is given by

(29) $$\rho_1 = a(a\sin\theta + s\cos\theta)(s\sin\theta - a\cos\theta)^{-1}.$$

When we substitute (20), (28) and (29) into the expression for the amplitude it becomes identical with (24).

As before, rays which are incident at Q_1 and encircle the sphere n times before emerging at P_1 are also diffracted rays. Since each of these rays passes through an odd number of foci the phase at P along the n^{th} number of this family is

(30) $$ik\left\{(\rho^2-a^2)^{1/2} + (r^2-a^2)^{1/2}\right\} + (ik-\alpha_m)\left[t_o + 2(n-1)\pi a\right] - i(2n+1)\pi/2\,.$$

Upon using (30), (24) and $A_o = (4\pi)^{-1}$ in (11) and summing over n from n = 1 to ∞

we obtain as the contribution to the field at P from all these rays which are tangent at Q_1

$$(31) \qquad v(r,\theta) = - \frac{i a^{1/2} \exp\left[ik\left\{(\rho^2-a^2)^{1/2}+(r^2-a^2)^{1/2}\right\}\right]}{4\pi(\rho r|\sin\theta|)^{1/2}(r^2-a^2)^{1/4}(\rho^2-a^2)^{1/4}} \sum_m D_m^2 \left\{1 + \exp\left[2\pi a(ik-\alpha_m)\right]\right\}^{-1}$$

$$\exp\left[(ik-\alpha_m)a\left\{2\pi-\theta-\cos^{-1}(a/r)-\cos^{-1}(a/\rho)\right\}\right] .$$

Since the diffracted rays through P which are initiated at Q_2 pass through a focus off the body (C_1), we must also include a phase decrease of $\pi/2$ when passing through this focus. Thus each member of this family of rays undergoes a phase decrease which is an even multiple of π. Upon taking this phase shift into consideration we find that the field due to the rays tangent at Q_2 is obtained from (31) by replacing θ by $-\theta$ and multiplying the resulting expression by $-i$. Thus the total diffracted field is

$$(32) \qquad u_d(r,\theta) = v(r,\theta) - i\, v(r,-\theta).$$

Again, by using (17) and (18), we see that (32) agrees with the diffracted field in (11) of Part II which was obtained by asymptotically expanding the solution of the corresponding boundary value problem.

7. Scalar diffraction by an arbitrary cylinder

In this section we treat scalar diffraction of an obliquely incident plane wave by a cylinder of arbitrary convex cross section. We will find that if the geometrical theory is valid for a normally incident plane wave, it is also valid for an obliquely incident one. To do this we first indicate the manner in which the corresponding boundary value problem at oblique incidence reduces to one at normal incidence. Then we show that the solution obtained by the geometrical theory reduces in exactly the same way.

First we introduce a rectangular coordinate system with the z-axis parallel to the generators of the cylinder. We orient the x-axis so that the incident plane wave, assumed to be of unit amplitude, is given by

$$U_i = e^{ik(z \cos \alpha + y \sin \alpha)} .$$

We also introduce a curvilinear orthogonal coordinate system (h_1,h_2) in the (x,y) plane with $h_1 = a$ being the equation of the generating curve of the cylinder.

Let us suppose that U is a solution of the following equations

$$(\nabla^2 + k^2)U = 0, \qquad h_1 \geq a$$

$$(\nabla^2 + k_1^2)U = 0, \qquad h_1 \leq a.$$

Furthermore we suppose that at $h_1 = a$, U satisfies one of the boundary conditions listed in Table I. The incident field U_i is given above and the scattered field $U-U_i$ must satisfy the appropriate two-dimensional radiation condition. Now we define $V(h_1,h_2)$ by setting $U(h_1,h_2,z) = V(h_1,h_2)e^{ikz \cos\alpha}$. Then it follows that V satisfies

$$(33) \qquad \begin{aligned} (\nabla_1^2 + k'^2)V &= 0, \qquad h_1 \geq a \\ (\nabla_1^2 + k_1'^2)V &= 0, \qquad h_1 \leq a. \end{aligned}$$

It also follows that V satisfies the same boundary conditions as does U at $h_1 = a$,[*] that $V_i = e^{ik'y}$ and that $V-V_i$ satisfies the two-dimensional radiation condition. In (33) $k' = k \sin \alpha$ and $k_1' = (k_1^2 - k^2 \cos^2\alpha)^{1/2}$, while ∇_1 is the two-dimensional Laplacian in the variables h_1 and h_2. These equations show that V is the two-dimensional field in a medium of propagation constant k' due to the wave $e^{ik'y}$ incident on a cylinder having propagation constant k_1'. From it we can find the diffracted field U by multiplying V by $e^{ikz \cos\alpha}$ and replacing k' by $k \sin \alpha$ and k_1' by $(k_1^2 - k^2\cos^2\alpha)^{1/2}$. We will now construct u and v, the fields given in these two problems by the geometrical theory. We will then show that u can be obtained from v in the same manner as U can be obtained from V. This demonstrates the consistency of the geometrical theory. It also permits the solution, by the geometrical theory, of a three-dimensional problem to be reduced to that of a two-dimensional one.

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[*] Care must be exercised, however, in Case III. Because k appears in the boundary condition, we must also replace Z by $Z/\sin \alpha$ in Table I.

To construct u let the cylinder be given by

$$\vec{x}(t,z) = (x(t), y(t), z). \tag{34}$$

Here $x(t)$ and $y(t)$ are such that $h_1\big(x(t),y(t)\big) = a$, and t is arc length along this curve. Then the geodesics on the cylinder are helices given by

$$\vec{x}(s) = \big(x(\rho s+\beta), y(\rho s+\beta), \gamma s+\delta\big). \tag{35}$$

The arc length along each helix is s if

$$\rho^2 + \gamma^2 = 1. \tag{36}$$

The constants ρ, β, γ, and δ depend upon the position and direction of the geodesic at $s = 0$. To evaluate them we let (x,y,z) be the coordinates of the point of observation and impose the condition that at $s = 0$ the tangent to the helix points in the direction opposite* to that through (x,y,z). This condition yields

$$x(\beta) - h\,\rho\,\dot{x}(\beta) = x \tag{37}$$

$$y(\beta) - h\,\rho\,\dot{y}(\beta) = y \tag{38}$$

$$\delta - h\,\gamma = z. \tag{39}$$

In these equations h is a constant of proportionality determining the distance along the tangent to the point of observation and the dot represents differentiation with respect to the argument.

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* We are assuming that the helix is parametrized so that the point $s = 0$ is the one at which the diffracted ray leaves the cylinder and proceeds to the point of observation. As s increases, the corresponding point moves backwards along the diffracted ray to its point of initiation.

We now denote by s_n the geodesic distance along the cylinder from the point of incidence to the point at which the diffracted ray leaves the surface. Then at $s = s_n$ the tangent to the helix must point in the direction opposite to the direction of incidence. This requires that

(40) $$\rho\,\dot{x}(\rho s_n + \beta) = 0$$

(41) $$\rho\,\dot{y}(\rho s_n + \beta) = -\sin\alpha$$

(42) $$\gamma = -\cos\alpha.$$

Since the argument of x and y is the arc length of the generating curve of the cylinder we have $\dot{x}^2 + \dot{y}^2 = 1$. Therefore, upon squaring and adding (37) and (38) we obtain

(43) $$h = r_\beta |\rho|^{-1}.$$

Here we have set $r_\beta = \left[\left(x(\beta)-x\right)^2 + \left(y(\beta)-y\right)^2\right]^{1/2}$. To analyze (40) we observe that the convexity of the generating curve ensures that there are precisely two points on the curve for which $\dot{x}(t) = 0$. Therefore, if t_1 and t_2 are two distinct points at which $\dot{x}(t) = 0$, and T denotes the length of the generating curve, (40) permits us to write

(44) $$\rho\, s_n^{(1)} + \beta = t_1 + nT$$

(45) $$\rho\, s_n^{(2)} + \beta = t_2 + nT.$$

From the fact that $\dot{x}^2 + \dot{y}^2 = 1$ we have $\dot{y} = \pm 1$ at t_1 and t_2. We let t_1 be such that $\dot{y}(t_1) = 1$ and then $\dot{y}(t_2) = -1$. Then from (41), the following two values of ρ correspond to $s_n^{(1)}$ and $s_n^{(2)}$ respectively

(46) $$\rho^{(1)} = -\sin\alpha$$

(47) $$\rho^{(2)} = \sin\alpha.$$

Upon substituting (46) or (47) into (43) we obtain

(48) $$h = r_\beta / \sin\alpha .$$

When this value of h and the above values of ρ are inserted into (37) and (38), these equations become

(49) $$x(\beta) \pm r_\beta \dot{x}(\beta) = x$$

(50) $$y(\beta) \pm r_\beta \dot{y}(\beta) = y.$$

These equations have two roots β_1 and β_2, corresponding to the upper and lower signs respectively*. The value of γ is given by (42) while from (39)

(51) $$\delta = z - r_\beta \cot\alpha .$$

Upon substituting the value of $\rho^{(1)}$ into (44) and the value of $\rho^{(2)}$ into (45) we obtain for each value of n,

(52) $$s_n^{(1)} = (\beta_1 - t_1 + nT) / \sin\alpha$$

(53) $$s_n^{(2)} = (-\beta_2 + t_2 + nT) / \sin\alpha .$$

The distance s appearing in (11) has two values, one for each family of rays, i.e., for each value of β. A simple calculation shows that these values are given by

(54) $$s_i = r_i / \sin\alpha \qquad i = 1, 2.$$

It is also easily seen that $a_g(s)$, the radius of curvature of the helix, is given in terms of $a_p(t)$, the radius of curvature of the generating curve, by

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* As a consequence the value of r_β depends upon the value of β. For $\beta = \beta_1$ we denote r_β by r_1, while for $\beta = \beta_2$ we denote r_β by r_2. When no confusion can occur we will continue to use r_β.

$$(55) \qquad a_g(s) = \frac{1}{\sin^2 \alpha} \, a_p(\rho s + \beta).$$

To complete the calculation of the geometrical quantities we need the value of the incident field at the points Q_n where the diffracted rays are initiated on the surface[*]. The values of U_i at the points Q_n are determined by substituting the values of y and z at the points of incidence into $e^{ik(y\sin\alpha + z\cos\alpha)}$. Making use of our previous calculation we obtain

$$(56) \qquad y\left(Q_n^{(i)}\right) = y\left(\rho^{(i)} s_n^{(i)} + \beta_i\right) = y(t_i + nT) = y(t_i)$$

$$(57) \qquad z\left(Q_n^{(i)}\right) = \gamma s_n^{(i)} + \delta^{(i)} = z - \left[r_i + nT + (-1)^i t_i - (-1)^i \beta_i\right] \cot \alpha \, .$$

Because of the cylindrical symmetry the quantity $d\sigma(Q)/d\sigma(P)$ in (11) is equal to 1, while $\rho_1 = \infty$. Consequently, taking into account all of the rays through P, (11) becomes

$$(58) \qquad u_d(P) = \sum_m \Bigg\{ D_m(P_1) s_1^{-1/2} \sum_{n=0}^{\infty} D_m\left(Q_n^{(1)}\right) U_i\left(Q_n^{(1)}\right) \exp\left[ik\left(s_n^{(1)} + s_1\right) - \int_0^{s_n^{(1)}} \alpha_m(s)\,ds\right]$$
$$+ D_m(P_2) s_2^{-1/2} \sum_{n=0}^{\infty} D_m\left(Q_n^{(2)}\right) U_i\left(Q_n^{(2)}\right) \exp\left[ik\left(s_n^{(2)} + s_2\right) - \int_0^{s_n^{(2)}} \alpha_m(s)\,ds\right] \Bigg\} \, .$$

Before substituting the values calculated for the quantities appearing in (58) we observe that $D_m\left(Q_n^{(i)}\right)$ is independent of n. This is because it depends only upon $a_g(s_n)$ which is clearly independent of n as we see from (55), (44) and (45).

---

[*] There are two families of diffracted rays, one corresponding to each value of β. The rays of one family are incident at the points $Q_n^{(1)}$ defined by (44). They all leave the cylinder at the point P_1 defined by (49) and (50) with $\beta = \beta_1$. The rays of the other family are incident at the points $Q_n^{(2)}$ defined by (45). They all leave the cylinder at the point P_2 defined by (49) and (50) with $\beta = \beta_2$.

Similarly, the function $\alpha_m(s)$ which also implicitly depends upon the points $Q_n^{(i)}$ at which the rays are initiated is also independent of n. Carrying out the substitutions and summing over n we obtain

$$(59)\qquad u_d(P) = (\sin\alpha)^{\frac{1}{2}} e^{ikz\cos\alpha} \sum_m D_m(P_1)D_m\left(Q^{(1)}\right)r_1^{-1/2}$$
$$\exp\left[ik\sin\alpha\left(\beta_1-t_1+r_1+y(t_1)\right) - \int_0^{(\beta_1-t_1)/\sin\alpha} \alpha_m\left(s,Q^{(1)}\right)ds\right]$$
$$\left(1 - \exp\left[ikT\sin\alpha - \int_0^{T/\sin\alpha} \alpha_m\left(s,Q^{(1)}\right)ds\right]\right)^{-1} + * \ .$$

In this result the field associated with the rays which leave the surface at P_1 has been written explicitly. The * denotes the corresponding field on the rays which leave the surface at P_2. The expression for this field can be obtained from the above sum by replacing the subscripts and superscripts 1 by 2 and changing the signs of β_1 and t_2 except in the argument of $y(t_2)$. The symbol $\alpha_m\left(s,Q^{(1)}\right)$ denotes the value of α_m at a point a distance s along a ray from any of the points $Q_n^{(1)}$ at which the rays are initiated. Similarly, the symbol $D_m\left(Q^{(1)}\right)$ denotes the value of D_m at any of the points $Q_n^{(1)}$. The values of α_m and D_m can be obtained from Table I.

Equation (59) is the diffracted field due to oblique incidence of a plane wave on a cylinder of arbitrary cross section. If we set $\alpha = \pi/2$ in this equation, we obtain the result for normal incidence previously obtained in [2]. Now if we replace k by k' and k_1 by k_1' in the result for normal incidence, we obtain the function v. We now wish to show that from v we can obtain u by replacing k' by $k \sin \alpha$, k_1' by $(k_1^2 - k^2\cos^2\alpha)^{1/2}$ and multiplying the result by $e^{ikz\cos\alpha}$. Upon performing these operations in (59), we find that u and v are related as above provided that

(60) $$\int_0^{W/\sin\alpha} \alpha_m(s;k,k_1)\,ds = \int_0^{W} \alpha_m\left(s;k\sin\alpha,\left[k_1^2-k^2\cos^2\alpha\right]^{1/2}\right)ds\ ;$$

(61) $$(\sin\alpha)^{1/4}\,D_m(A;k,k_1) = D_m\left(A;k\sin\alpha,\left[k_1^2-k^2\cos^2\alpha\right]^{1/2}\right)\ .$$

In (61) A denotes an arbitrary point while in (60) W denotes an arbitrary upper limit of integration. The dependence of D_m and of α_m on k and k_1 has been indicated explicitly.

To verify (60) we express the value of $a = a_g(s)$ appearing in the expression for α_m in Table I in terms of a_p by means of equation (55) to obtain

(62) $$\alpha_m(s;k,k_1) = e^{-i\pi/6}\left(\frac{k}{6a_p^2(\rho s+\beta)}\right)^{1/3} q_m \sin^{4/3}\alpha\ .$$

Upon substituting (62) into the left-hand side of (60) and changing variables by setting $s\sin\alpha = \xi$ we obtain

(63) $$\int_0^{W/\sin\alpha} \alpha_m(s;k,k_1)\,ds = e^{-i\pi/6}\left(\frac{k\sin\alpha}{6}\right)^{1/3}\int_0^{W}\frac{q_m\,d\xi}{a_p^{2/3}(\pm\xi+\beta)}\ .$$

In (63) we have used the fact that $\rho = \pm\sin\alpha$.

Now in Cases I and II, where q_m is a numerical constant, equation (63) demonstrates the validity of (60). In Cases III and IV, however, q_m is a function of k, k_1, and a. Therefore, the right-hand sides of (63) and (60) will be identical if

(64) $$q_m(k,k_1,a_g) = q_m\left(k\sin\alpha,\left[k_1^2-k^2\cos^2\alpha\right]^{1/2},a_p\right)\ .$$

In Case III, when q_m is also a function of Z, the Z contained in the right hand side of (64) must be replaced by $Z/\sin\alpha$ (see footnote, p. 21). When use is made of (55), together with the entries for $A'(q_m)/A(q_m)$ in Table I, (64) is easily verified.

To verify (61) we first use Table I to write $D_m(A;k,k_1)$ in the form

$$D_m(A;k,k_1) = k^{-1/12} a^{1/6} f(q_m). \tag{65}$$

In (65), $f(q_m)$ is in all four cases a function only of the variable $q_m = q_m(k,k_1,a_g)$. When use is made of (55) and (64), (65) is readily verified.

Thus we have shown that the result (59), given by the geometrical theory for diffraction by an obliquely incident wave, can be obtained from the result for a normally incident wave. The relation between these two fields is the same as for the solutions of the corresponding boundary value problems considered at the beginning of this section.

8. Electromagnetic diffraction by a sphere

Let us now consider the diffraction of a linearly polarized plane electromagnetic wave by a sphere. We will use the same notation as in Section 6 and assume that the incident electric field is parallel to the x-axis and propagating in the positive z direction. Then the incident electric field can be written as

$$E_x^i = Ae^{ikz} \qquad E_y^i = E_z^i = 0. \tag{66}$$

The incident rays are parallel to the z-axis so the only points of tangency will lie on the "equator" of the sphere in the plane $z = 0$. The diffracted rays are plane curves (great circles) on the sphere. Therefore, the rays through a point P with coordinates (r,θ,ϕ) will lie in the plane ϕ = constant which passes through P. This plane is shown in Figure 7. There are two families of diffracted rays through P. One is initiated at $Q_1(a,\pi/2,\phi)$ and leaves the sphere at $P_1\left(a,\theta+\cos^{-1}(a/r),\phi\right)$ provided $\theta+\cos^{-1}(a/r) < \pi$. The other family originates at $Q_2(a,\pi/2,\phi+\pi)$ and leaves the sphere at $P_2\left(a,\theta-\cos^{-1}(a/r),\phi\right)$ provided $\theta > \cos^{-1}(a/r)$. If $\theta+\cos^{-1}(a/r) > \pi$ then P_1 has coordinates $\left(a,2\pi-\theta-\cos^{-1}(a/r),\phi+\pi\right)$. If $\theta < \cos^{-1}(a/r)$ then P_2 has coordinates $\left(a,\cos^{-1}(a/r)-\theta,\phi+\pi\right)$. The rays of each family encircle the sphere any number of times before leaving it.

We will calculate the diffracted field $E^d(P)$ by using (19). Therefore, we first calculate the rectangular components of the d vectors at the points Q_1 and Q_2.

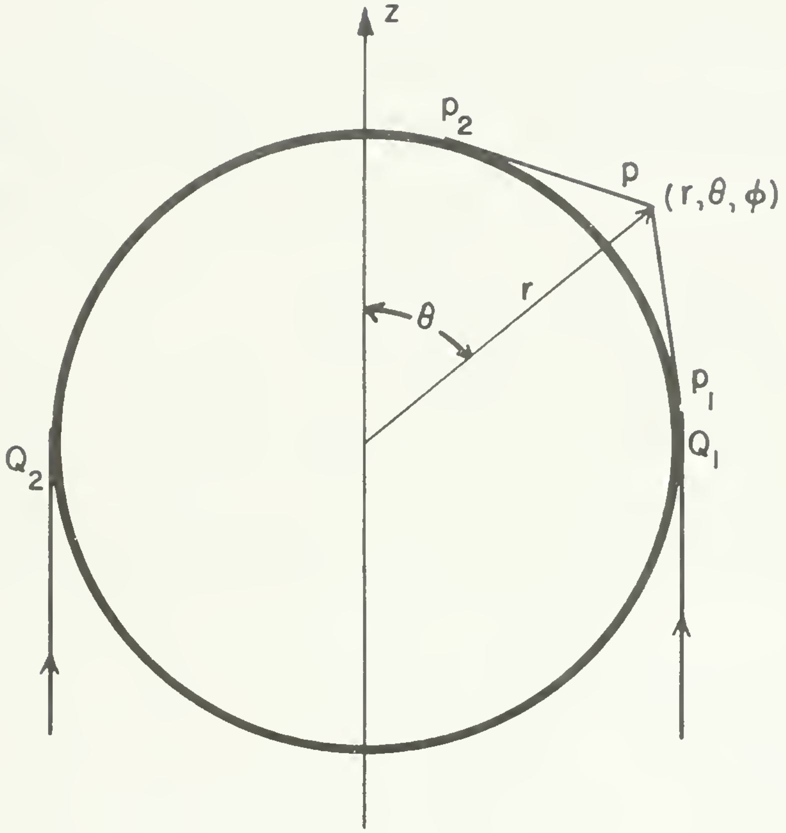


Figure 7: The ray system associated with the diffraction of an electromagnetic wave by a sphere. The diffracted rays are incident on the equator at the points Q_1 and Q_2 and leave tangentially at the points P_1 and P_2.

These vectors are

$$(67)\quad \vec{d}_1(Q_1) = (0,0,1),\ \vec{d}_2(Q_1) = (\cos\phi,\sin\phi,0),\ \ \vec{d}_3(Q_1) = (-\sin\phi,\cos\phi,0)$$

$$(68)\quad \vec{d}_1(Q_2) = (0,0,1),\ \vec{d}_2(Q_2) = (-\cos\phi,-\sin\phi,0),\ \vec{d}_3(Q_2) = (\sin\phi,-\cos\phi,0).$$

Similarly, the spherical components of the corresponding vectors at the points P_1 and P_2 are

$$(69)\quad \begin{aligned} &\vec{d}_1(P_1) = \left(r^{-1}(r^2-a^2)^{1/2},-a/r,0\right),\ \vec{d}_2(P_1) = \left(a/r,r^{-1}(r^2-a^2)^{1/2},0\right),\\ &\qquad\qquad\qquad \vec{d}_3(P_1) = (0,0,1)\\ &\vec{d}_1(P_2) = \left(r^{-1}(r^2-a^2)^{1/2},a/r,0\right),\ \vec{d}_2(P_2) = \left(a/r,-r^{-1}(r^2-a^2)^{1/2},0\right),\\ &\qquad\qquad\qquad \vec{d}_3(P_2) = (0,0,-1)\ . \end{aligned}$$

From (67) and (68) and the expression (66) for the incident field, we find that at Q_1 and Q_2 this field has the components

$$(70)\quad \begin{aligned} E_2^i(Q_1) &= A\cos\phi, & E_2^i(Q_2) &= -A\cos\phi,\\ E_3^i(Q_1) &= -A\sin\phi, & E_3^i(Q_2) &= A\sin\phi. \end{aligned}$$

By using the methods of Section 6 it can easily be seen that the function $v(P,Q_1)$ appearing in (19) is identical with the function $v(r,\theta)$ defined by (40) of Part II. Similarly, $v(P,Q_2)$ is equal to $e^{-i\pi/2}v(r,-\theta)$; while $w(P,Q_1)$ is equal to the function $u(r,\theta)$ also defined by (40) and $w(P,Q_2)$ is equal to $e^{-i\pi/2}u(r,-\theta)$.

Upon using the above together with (70) in (19) we obtain

$$(71)\quad \begin{aligned} E^d(P) = A\cos\phi&\left[\vec{d}_2(P_1)v(r,\theta) - e^{-i\pi/2}\,\vec{d}_2(P_2)v(r,-\theta)\right]\\ &- A\sin\phi\left[\vec{d}_3(P_1)u(r,\theta) - e^{-i\pi/2}\,\vec{d}_3(P_2)u(r,-\theta)\right]. \end{aligned}$$

If we now express the d vectors in terms of their spherical components as given by (69) we see that (71) is in agreement with (41), (42), and (45) of Part II which was obtained by asymptotically expanding the solution to the corresponding boundary value problem.

9. Electromagnetic diffraction by an arbitrary cylinder

We now consider the diffraction of a plane linearly polarized electromagnetic wave by a cylinder of arbitrary convex cross section. We introduce a rectangular coordinate system as in Section 7 so that the direction of incidence is given by the unit vector $\overrightarrow{d_1^o} = (0, \sin\alpha, \cos\alpha)$. If the electric vector makes an angle $(\pi/2 - \theta)$ with the (x,y) plane, then the rectangular components of the incident electric field can be written as

$$\begin{aligned} E_x^i &= A \sin\theta\, e^{ik(y\sin\alpha + z\cos\alpha)} \\ E_y^i &= A \cos\theta \cos\alpha\, e^{ik(y\sin\alpha + z\cos\alpha)} \qquad (72) \\ E_z^i &= -A \cos\theta \sin\alpha\, e^{ik(y\sin\alpha + z\cos\alpha)}. \end{aligned}$$

The diffracted ray system has been described in Section 7. At a given point P it consists of two families of rays. The rays of one family are initiated at points $Q_n^{(1)}$ and leave the cylinder at the point P_1, while the rays of the other family are initiated at points $Q_n^{(2)}$ and leave the cylinder at the point P_2.

In order to calculate the diffracted field by means of (19) we first calculate the rectangular components of the d vectors at the points $Q_n^{(1)}$ and $Q_n^{(2)}$. These vectors are independent of n and are given by

$$\begin{aligned} &\vec{d}_1(Q_n^1) = (0, \sin\alpha, \cos\alpha), \quad \vec{d}_2(Q_n^1) = (1,0,0), \quad \vec{d}_3(Q_n^1) = (0, \cos\alpha, -\sin\alpha) \\ (73)\quad &\vec{d}_1(Q_n^2) = (0, \sin\alpha, \cos\alpha), \quad \vec{d}_2(Q_n^2) = (-1,0,0), \quad \vec{d}_3(Q_n^2) = (0, -\cos\alpha, \sin\alpha). \end{aligned}$$

Similarly, in the notation of Section 7, the rectangular components of the corresponding vectors at the points P_1 and P_2 are

$$\vec{d}_1(P_1) = \left(\dot{x}(\beta_1)\sin\alpha,\ \dot{y}(\beta_1)\sin\alpha,\ \cos\alpha\right),\quad \vec{d}_2(P_1) = -a_p(\beta_1)\left(\ddot{x}(\beta_1),\ \ddot{y}(\beta_1),\ 0\right)$$

$$\vec{d}_3(P_1) = \left(a_p(\beta_1)\ddot{y}(\beta_1)\cos\alpha,\ -a_p(\beta_1)\,\ddot{x}\,(\beta_1)\cos\alpha,\ -\sin\alpha\right)$$

(74)

$$\vec{d}_1(P_2) = \left(-\dot{x}(\beta_2)\sin\alpha,\ -\dot{y}(\beta_2)\sin\alpha,\ \cos\alpha\right),\quad \vec{d}_2(P_2) = -a_p(\beta_2)\left(\ddot{x}\,(\beta_2),\ \ddot{y}(\beta_2),\ 0\right)$$

$$\vec{d}_3(P_2) = \left(a_p(\beta_2)\,\ddot{y}(\beta_2)\cos\alpha,\ -a_p(\beta_2)\,\ddot{x}(\beta_2)\cos\alpha,\ \sin\alpha\right).$$

By using (72) and (73) we find that the components of E^i at the points $Q_n^{(1)}$ and $Q_n^{(2)}$ are

$$E_1^i(Q_n^{(1)}) = 0 \quad , \quad E_2^i(Q_n^{(1)}) = A\sin\theta\, e^{ik(y\sin\alpha\, +\, z\cos\alpha)}$$

$$E_3^i(Q_n^{(1)}) = A\cos\theta\, e^{ik(y\sin\alpha\, +\, z\cos\alpha)} \tag{75}$$

$$E_1^i(Q_n^{(2)}) = 0 \quad , \quad E_2^i(Q_n^{(2)}) = -A\sin\theta\, e^{ik(y\sin\alpha\, +\, z\cos\alpha)}$$

$$E_3^i(Q_n^{(2)}) = -A\cos\theta\, e^{ik(y\sin\alpha\, +\, z\cos\alpha)}. \tag{76}$$

In (75), y and z are to be evaluated at $Q_n^{(1)}$, while in (76) y and z are to be evaluated at $Q_n^{(2)}$. The values of y and z at the points Q_n are given by (56) and (57) of Section 7.

In the present notation (19) becomes

$$E^d(P) = \sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[E_2^i(Q_n^{(j)})\ v\,(P,Q_n^{(j)})\,\vec{d}_2(P_j) + E_3^i(Q_n^{(j)})\ w\,(P,Q_n^{(j)})\vec{d}_3(P_j)\right]. \tag{77}$$

The calculations of Section 7 show that

$$(78)\qquad \left.\begin{aligned} \sum_{n=0}^{\infty} E_2^i(Q_n^{(j)})v(P,Q_n^{(j)}) &= v_j^d(P) = (-1)^{j+1}A\sin\theta \\ \sum_{n=0}^{\infty} E_3^i(Q_n^{(j)})w(P,Q_n^{(j)}) &= w_j^d(P) = (-1)^{j+1}A\cos\theta \end{aligned}\right\} (\sin\alpha)^{\frac{1}{2}} e^{ikz\cos\alpha}$$

$$\sum_m D_m(P_j)D_m(Q^{(j)})r_j^{-1/2}\exp\left[ik\sin\alpha\left(r_j+(-1)^jt_j-(-1)^j\beta_j+y(t_j)\right)-\int_0^{[(-1)^jt_j-(-1)^j\beta_j]/\sin\alpha}\alpha_m(s,Q^{(j)})ds\right]$$

$$\left(1-\exp\left[ikT\sin\alpha-\int_0^{T/\sin\alpha}\alpha_m(s,Q^{(j)})ds\right]\right)^{-1} .$$

The values of D_m and α_m in the expression for $v_j^d(P)$ are those appropriate for the normal component of the electric field, while the values of these quantities appearing in the expression for $w_j^d(P)$ are those appropriate for the tangential component of the electric field. In Section 5 we have described the way in which quantities are to be determined.

Upon using (74), (77), and (78) we obtain for the rectangular components of the diffracted field

$$(79)\qquad \begin{aligned} E_x^d(P) &= \sum_{j=1}^{2}\left[-v_j^d(P)a_p(\beta_j)\,\dot{x}(\beta_j)+w_j^d(P)a_p(\beta_j)\,\dot{y}(\beta_j)\cos\alpha\right] \\ E_y^d(P) &= \sum_{j=1}^{2}\left[-v_j^d(P)a_p(\beta_j)\,\dot{y}(\beta_j)-w_j^d(P)a_p(\beta_j)\,\dot{x}(\beta_j)\cos\alpha\right] \\ E_z^d(P) &= -\sum_{j=1}^{2}(-1)^j\,w_j^d(P)\sin\alpha \quad . \end{aligned}$$

Let us now specialize (79) to the case of a point $P(\rho,\phi,z)$ in the shadow of an imperfectly conducting circular cylinder of radius a. In this case the function $x(t)$ is equal to $a \cos(t/a)$, while $y(t)$ is equal to $a \sin(t/a)$. Simple calculations show that $\beta_1 = a(\phi - \tau)$ and $\beta_2 = a(\phi + \tau)$ where $\tau = \cos^{-1}(a/\rho)$, and $0 < \tau < \pi/2$. The quantity r_j is equal to $(\rho^2 - a^2)^{1/2}$, while $t_1 = 0$ and $t_2 = a\pi$. The radius of curvature a_p is a, while the torsion τ which enters into the determination of α_m and D_m is equal to $a^{-1} \sin\alpha \cos\alpha$.

In order to identify (79) with the asymptotic expansion of the exact solution we first determine the values of α_m and D_m appearing in (78). From the results of Section 5 we see that for the function $v_j^d(P)$ the values of q_m are determined by the equation

$$(80) \qquad \frac{A'(q_m)}{A(q_m)} = e^{\frac{5\pi i}{6}} \left(\frac{k a_g}{6}\right)^{\frac{1}{3}} \left[\left(\frac{k_1}{k}\right)^2 - 1\right]^{\frac{1}{2}} \frac{k^2 \mu_1}{\mu} \left\{(k_1^2 - k^2)(a_g \tau)^2 + k_1^2\right\}^{-1} ,$$

where we have denoted the radius of curvature of the diffracted ray on the surface by a_g in order to avoid confusion with the radius a of the cylinder. Upon substituting $a(\sin^2\alpha)^{-1}$ for a_g and $a^{-1}\sin\alpha \cos\alpha$ for τ, we may write (80) as

$$(81) \qquad \frac{A'(q_m)}{A(q_m)} = e^{\frac{5\pi i}{6}} \left(\frac{k' a}{6}\right)^{\frac{1}{3}} \left[\left(\frac{k_1'}{k'}\right)^2 - 1\right]^{\frac{1}{2}} \frac{k'^2 \mu_1}{\mu k_1'^2}$$

where we have introduced $k' = k \sin\alpha$ and $k_1'^2 = k_1^2 - k^2 \cos^2\alpha$. From (81) we see that q_m entering into the determination of $v_j^d(P)$ are identical to the q_m entering into the determination of the function $v(\rho, \phi)$ of Part II, Section 4. In fact, upon comparing (78) and (24) of Part II we have

$$(82) \qquad \begin{aligned} v_1^d(P) &= A \sin\theta \, e^{ikz\cos\alpha} v(\rho, \phi) \\ v_2^d(P) &= -A \sin\theta \, e^{ikz\cos\alpha} v(\rho, \pi - \phi). \end{aligned}$$

In a similar way we conclude that

$$
\begin{aligned}
w_1^d(P) &= A \cos \theta \, e^{ikz\cos\alpha} \, w(\rho,\phi) \\
w_2^d(P) &= -A \cos \theta \, e^{ikz\cos\alpha} \, w(\rho,\pi-\phi),
\end{aligned}
\tag{83}
$$

where $w(\rho,\phi)$ is the function defined by (24) of Part II.

Upon substituting (82) and (83) into (79) we obtain

$$
\begin{aligned}
E_x^{d'}(P) = A\, a \sin\theta \Big\{ & \rho^{-1}\cos\phi \big[v(\rho,\phi) - v(\rho,\pi-\phi)\big] \\
& + (a\rho)^{-1}(\rho^2-a^2)^{1/2}\sin\phi \big[v(\rho,\phi) + v(\rho,\pi-\phi)\big] \Big\} \\
& - A a\cos\theta\cos\alpha \Big\{ \rho^{-1}\sin\phi \big[w(\rho,\phi) - w(\rho,\pi-\phi)\big] \\
& - (a\rho)^{-1}(\rho^2-a^2)^{1/2}\cos\phi \big[w(\rho,\phi) + w(\rho,\pi-\phi)\big] \Big\}
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
F_y^{d'}(P) = A\, a \sin\theta \Big\{ & \rho^{-1}\sin\phi \big[v(\rho,\phi) - v(\rho,\pi-\phi)\big] \\
& - (a\rho)^{-1}(\rho^2-a^2)^{1/2}\cos\phi \big[v(\rho,\phi) + v(\rho,\pi-\phi)\big] \Big\} \\
& + A a\cos\theta\cos\alpha \Big\{ \rho^{-1}\cos\phi \big[w(\rho,\phi) - w(\rho,\pi-\phi)\big] \\
& + (a\rho)^{-1}(\rho^2-a^2)^{1/2}\sin\phi \big[w(\rho,\phi) + w(\rho,\pi-\phi)\big] \Big\}
\end{aligned}
\tag{85}
$$

$$
E_z^{d'}(P) = -A \cos\theta \sin\alpha \, \big(w(\rho,\phi) + w(\rho,\pi-\phi) \big) \, . \tag{86}
$$

The expressions for the components of the electric field are primed to indicate the supression of a common $e^{ikz\cos\alpha}$ factor. If we now use the fact that the cylindrical components of E are given by

$$E_r^{d'} = E_x^{d'} \cos \phi + E_y^{d'} \sin \phi$$

$$E_\phi^{d'} = -E_x^{d'} \sin \phi + E_y^{d'} \cos \phi$$

we see that (84), (85) and (86) agree with the sum of (25) and (26) of Part II which is the asymptotic expansion of the exact solution of the vector boundary value problem.

10. The field near caustics

The caustic of a given family of rays is defined as that surface or curve which is everywhere tangent to the ray system. Thus, the surface of a diffracting body is a caustic of the family of diffracted rays resulting from a given field being incident on the body. In addition to the diffracting bodies in the problems we have considered an axial caustic (the polar axis) is formed in the sphere problems.

In accordance with optical experience and as we see from equations (11) and (24) on a caustic the field constructed by the present method becomes infinite. A finite value for the field on the caustic and a corrected value for the field near the caustic can be obtained by a slight modification of our method.

Let us first correct equation (11) to hold on or near the body. To do this we make use of a canonical problem, the cylinder. Equation (5) of Part II was obtained from (4) by using the Debye expansion [7] of the Hankel functions $H_{\nu_\ell}^{(1)}(kr)$ and $H_{\nu_\ell}^{(1)}(k\rho)$ appearing. However, when we are near or on the body $r \sim a$ and since $\nu_\ell \sim ka$ this expansion is not appropriate. In this case a valid approximation is the Hankel expansion [8] of the Hankel Function. Thus, we can modify (5) to hold near or on* the body if we multiply it by the ratio of the

---

* To be precise (5) represents the leading term in the asymptotic expansion of the field as long as $r \neq a$. However, when r is near a this leading term has no value for practical computational purposes.

Hankel expansion of $H^{(1)}_{\nu_\ell}(kr)$ to the Debye expansion. Equation (87) below gives the ratio R_ℓ of the Hankel to the Debye expansion for r near a.

$$(87) \qquad R_\ell = 2^{1/2}6^{1/3}\pi^{-1/2}k^{1/6}(r^2-a^2)^{1/4}\exp\left[-ik(r^2-a^2)^{1/2} + i\nu_\ell \cos^{-1}(a/r) - \frac{i\pi}{12}\right] \cdot A\left(\left\{k(a-r) + q_\ell (ka)^{1/3}\right\}(kr)^{-1/3}\right) .$$

We now postulate that on or near the body (11) should be modified by multiplying by R_ℓ . This requires that the quantities r and a appearing in the above be given meaning for an arbitrary convex body. The radius a we interpret in the same way as we did in the expressions for α_m and D_m. To interpret r notice that $s = (r^2-a^2)^{1/2}$ so that if we set $r = (s^2+a^2)^{1/2}$ then r is defined in terms of geometrical quantities.

Notice also that in case the source is located on the body we can modify equation (5) so that it remains valid for $\rho = a$ by multiplying it by R_ℓ with r replaced by ρ. This admits of an obvious generalization to a convex body on which a point source is located.

The derivation of the asymptotic expansion of the field on an axial caustic has been previously studied [3]. The method consists of writing down an exact solution of the reduced wave equation which possesses an axial caustic and then using the exact solution to evaluate the field on the caustic.

In cylindrical coordinates $(x,\rho,\emptyset)$ the function

$$(88) \qquad u = A\, e^{ikx\sin\delta} J_n(k\rho \cos\delta) \cos n\emptyset$$

is an exact solution of the wave equation for arbitrary δ and A. If we then asymptotically expand the integral-ordered Bessel function in (88) we can recognize (88) as describing a family of rays which possesses the x axis as a caustic and makes an angle of $\pi/2 - \delta$ with it. By then comparing (88) with its asymptotic expansion it can be easily shown that to correct a given optical field which possesses, i.e., becomes infinite on, an axial caustic we must multiply the expression for the field by

(89) $$(1/2)(2\pi k\rho\cos\delta)^{1/2}\sec\left[k\rho\cos\delta - (n+\tfrac{1}{2})\tfrac{\pi}{2}\right] J_n(k\rho\cos\delta) .$$

Fquation (89) gives the correction factor on or near any axial caustic. If we are on the caustic ($\rho = 0$) then we must multiply the optical field by (89) and then let ρ tend to zero.

Let us now apply (89) to the case of back scattering from a sphere. In this case $\rho = r\sin\theta$, $\cos\delta = a/r$, $n = 0$, and (89) becomes

(90) $$\frac{(2\pi ka\sin\theta)^{1/2}}{2\cos\left(\frac{\pi}{4}\right)} .$$

Thus for $\theta = 0$, equation (11) of Part II becomes

(91) $$u(P) \sim u_g(P) + \lim_{\theta\to 0}\frac{(2\pi ka\sin\theta)^{1/2}}{2\cos\left(\frac{\pi}{4}\right)} u_d(P).$$

As an example of the numerical usefulness of the geometric theory we use (91) to compute the back scattering from a rigid sphere upon which a plane acoustic wave, $u_i = e^{ikz}$, is incident and compare the result with a computation based upon summing the exact series solution. For the geometrically reflected wave $u^g(P)$ we use the leading term of a previously calculated asymptotic expansion[5]

(92) $$u_g(P) \sim -a(2r)^{-1}\exp\left[ik(r-2a)\right] .$$

Equation (92) is the far field result, i.e., for r tending to infinity.

If we follow the method of Section 6 for the case of an incident plane wave we obtain for points in the lit region, but not on the axis

(93) $$u_d(P) \sim w(r,\theta) - iw(r,-\theta).$$

In (93)

$$(94) \qquad w(r,\theta) = -\frac{ia^{1/2}\exp\left[ik(r^2-a^2)^{1/2}\right]}{(r\sin\theta)^{1/2}(r^2-a^2)^{1/4}} \sum D_m^2 \left\{1+\exp\left[2\pi a(ik-\alpha_m)\right]\right\}^{-1}$$

$$\exp\left[(ik-\alpha_m)a\left\{\frac{3\pi}{2} - \theta - \cos^{-1}(a/r)\right\}\right] \quad .$$

The values of D_m and α_m in (94) are given in Table I, Case II. In the following we will use only the mode corresponding to $m = 0$. When we substitute (92) and (95) into (91) and let r tend to infinity we obtain

$$(95) \qquad u(P) \sim -a(2r)^{-1} \exp\left[ik(r-2a)\right] F(ka)$$

where

$$(96) \qquad F(ka) = 1 - 2\pi^2 6^{-1/3} q_o^{-1} A^{-2}(q_o)(ka)^{1/3} \exp\left[-\pi q_o 6^{-1/3}(ka)^{1/3} \sin\frac{\pi}{3}\right]$$

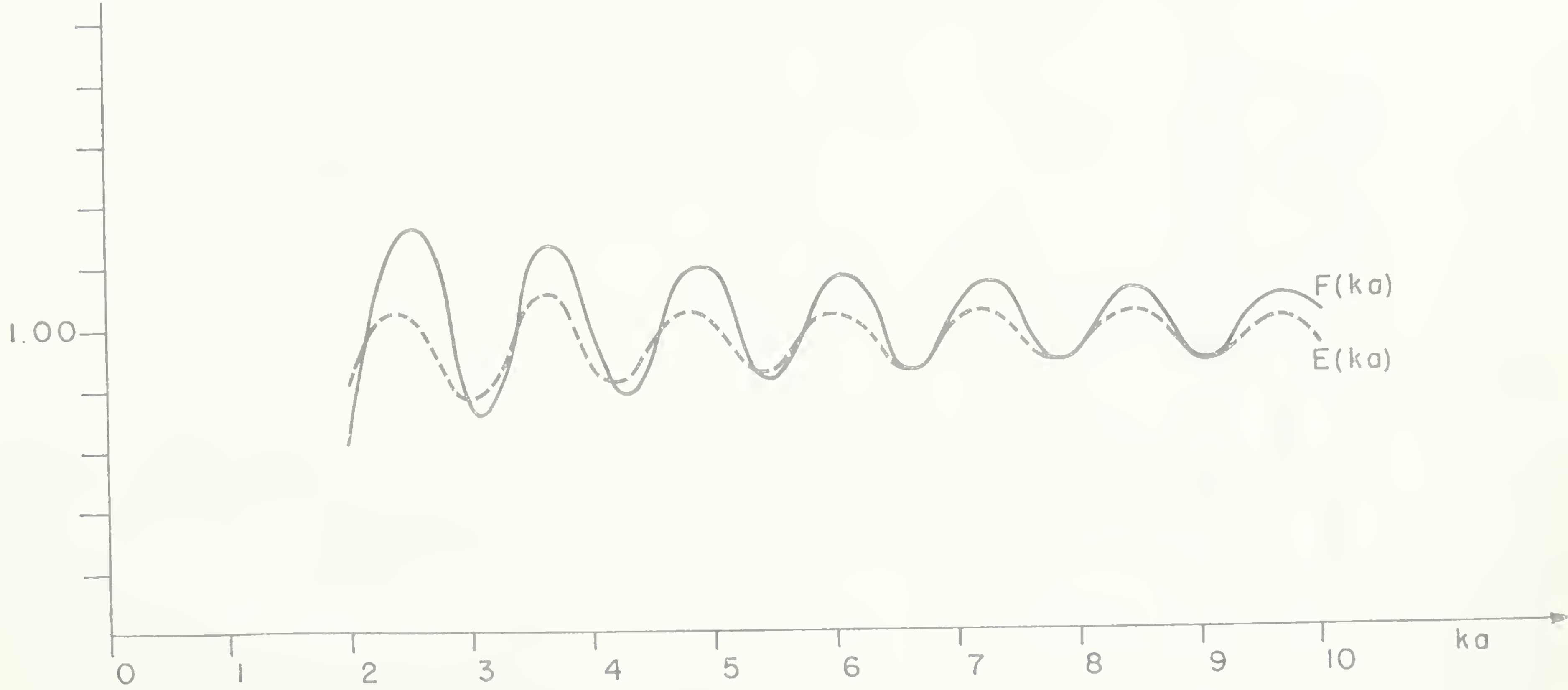
$$\exp\left[\frac{i\pi}{3} + ika(\pi+2) + i\pi q_o 6^{-1/3}(ka)^{1/3}\cos\frac{\pi}{3}\right] \quad .$$

By an analysis of the exact series solution to the associated boundary value problem it is easy to see that $|F(ka)|^2$ is the asymptotic expansion of the series

$$(97) \qquad |E(ka)|^2 = \frac{4}{(ka)^2} \left| \sum_{n=o}^{\infty} (-1)^n (2n+1) \frac{j_n(ka)}{h_n^{(1)\prime}(ka)} \right|^2 \quad .$$

In Figure 8 the functions $|E(ka)|$ and $|F(ka)|$ are graphed for $2 \leq ka \leq 10$.

Figure 8: The graphs of E(ka) and F(ka) arising in the problem of back scattering from a sphere. The function E(ka) is shown dashed. The straight line parallel to the axis with ordinate 1 is the classical geometrical optics limit. The function F(ka) is obtained from the geometrical theory of diffraction and E(ka) is obtained by asymptotically expanding the exact solution to the corresponding boundary value problem.



Appendix I

In order to apply equation (11) for $u_d(P)$ to the sphere it is necessary to calculate the radius of curvature ρ. This quantity was defined as the non-zero principal radius of curvature of the wave front on the body. Now the radius of curvature of a plane curve is equal to the distance from the curve along the normal to the point where neighboring normals intersect. Furthermore, the rays are normals to the wavefronts. Therefore, the radius of curvature of the normal section* cut out of the wavefront through P_1 or P_2 by the plane of Figure 5 is zero. Since the principal directions are orthogonal, ρ_1 is the radius of curvature of the normal section obtained by intersecting the wavefront with a plane π perpendicular to the plane of Figure 5.

Let us first calculate $\rho_1(P_1)$. The trace of the wavefront on the plane π is simply the point P_1. However, ρ_1 was defined as the limiting value of the principal radius curvature of the normal section of the wavefront through a point P_1' on the diffracted ray as P_1' tends to P_1. If we again use the fact that the radius curvature is the distance from the curve along the normal to the point where neighboring normals intersect, it is evident that the limiting value of ρ_1 is as shown in Figure 5. This is the distance along a tangent to the sphere through the point P_1 from P_1 to the point of intersection of the tangent and the polar axis. A simple geometric calculation then yields (23) for $\rho_1(P_1)$.

---

* A normal section of a surface is the plane curve cut out of the surface by a plane containing the normal to the surface. Since the rays are orthogonal to the wavefront the normal sections through P_1 or P_2 are planes cut through P_1 or P_2 containing the diffracted rays through these points.

Part II. Asymptotic Expansions of Exact Solutions of Diffraction Problems

1. Introduction

In Part I we solved diffraction problems by the geometrical theory of diffraction. Now, instead, we consider such problems as boundary value problems, the field being required to satisfy one or more partial differential equations. We consider special cases which can be solved by the method of separation of variables. The solutions, which are well-known infinite series of product solutions, are useful only for small values of ka. By utilizing an appropriate transformation, due to G. N. Watson,* we are able to evaluate these solutions asymptotically for large values of ka. In some of the cases which we treat this evaluation has been carried out before for special boundary conditions. In others, such as that of the finitely conducting circular cylinder hit obliquely by an electromagnetic wave, the asymptotic evaluation has apparently not been done before.

In Part I the results of the geometrical theory of diffraction have been shown to agree with the present results.

2. Scalar diffraction by a circular cylinder

In this section we will formulate and solve the boundary value problem of diffraction of a cylindrical wave by a circular cylinder of radius a. Then we will expand the solution asymptotically for large values of ka. We follow the method of W. Franz [9], which we apply to various cases not previously treated.

We consider a scalar field $u(r,\theta)$ which satisfies the equations

$$(\nabla^2 + k^2)u = -r^{-1}\delta(r-\rho)\delta(\theta) \qquad r \geq a$$

$$(\nabla^2 + k_1^2)u = 0 \qquad r \leq a$$

$$\lim_{r\to\infty} r^{1/2}(u_r - i\,ku) = 0 \,.$$

---

* A description of the Watson transformation with particular reference to the early literature is given in the introduction of [8].

We will consider four different kinds of boundary conditions at $r = a$. They correspond to different physical problems and are listed below. In the first three cases the second equation above, which applies for $r \leq a$, is to be omitted.

		Physical Description	
Case	Boundary Condition	Acoustic	Electromagnetic
I	$u = 0$	u = pressure soft cylinder	$u = E_z$ perfectly conducting cylinder
II	$u_r = 0$	u = pressure hard cylinder	$u = H_z$ perfectly conducting cylinder
III	$u_r = -ikZu$	u = pressure absorbing cylinder	$u = E_z$ or H_z absorbing cylinder
IV	u_r continuous $[bu]_{r=a_+} = [b_1u]_{r=a_-}$	bu = pressure $r > a$ b_1u = pressure $r < a$ $\frac{b}{b_1} = \frac{\text{density of medium}}{\text{density of cylinder}}$	$u = \mu^{-1}E_z$ $r > a$ $u = \mu_1^{-1}E_z$ $r < a$ $\frac{b}{b_1} = \frac{\mu}{\mu_1}$ or $u = \mu k^{-2}H_z$ $r > a$ $u = \mu k_1^{-2}H_z$ $r < a$ $\frac{b}{b_1} = \frac{k^2\mu_1}{k_1^2\mu}$

By expanding the diffracted field in cylindrical wave functions and imposing the boundary conditions we find

$$(1) \quad u = \frac{i}{4}\sum_{m=-\infty}^{+\infty} e^{im\theta}\left[J_m(kr_<) - \frac{\Omega J_m(ka)}{\Omega H_m^{(1)}(ka)} H_m^{(1)}(kr_<)\right] H_m^{(1)}(kr_>) \ , \quad r > a \ .$$

Here $r_>$ and $r_<$ are respectively the larger and smaller of the quantities r and ρ.

The operator Ω , which is determined by the boundary conditions, is defined in the following table

Case I: $\Omega F(x) = F(x)$

Case II: $\Omega F(x) = \frac{dF(x)}{dx}$

Case III: $\Omega F(x) = \frac{d}{dx} F(x) + iZ\, F(x)$

Case IV: $\Omega F(x) = b\, k_1\, J_m'(k_1 a)F(x) - b_1 k\, J_m(k_1 a)\, \frac{d}{dx} F(x)$.

In Case IV, F(x) denotes a cylinder function of order m.

The representation of the field as given by equation (1) is useful only for small values of ka. A numerically useful result for large values of ka is obtained by employing the well-known Watson transformation which leads to an asymptotic expansion of the field. Letting C be a contour encircling the real axis as shown in Figure 9 , we can write the sum in (1) as the following contour integral

$$(2) \qquad u = -\frac{1}{8}\int_C \frac{e^{i\nu(\theta-\pi)}}{\sin \nu\pi}\, \frac{H_\nu^{(1)}(kr_>)}{\Omega H_\nu^{(1)}(ka)}\left[\Omega H_\nu^{(1)}(ka) J_\nu(kr_<) - \Omega J_\nu(ka) H_\nu^{(1)}(kr_<)\right] d\nu \ .$$

When ν is replaced by $-\nu$ on that part of the contour C for which $\mathrm{Im}\,\nu < 0$

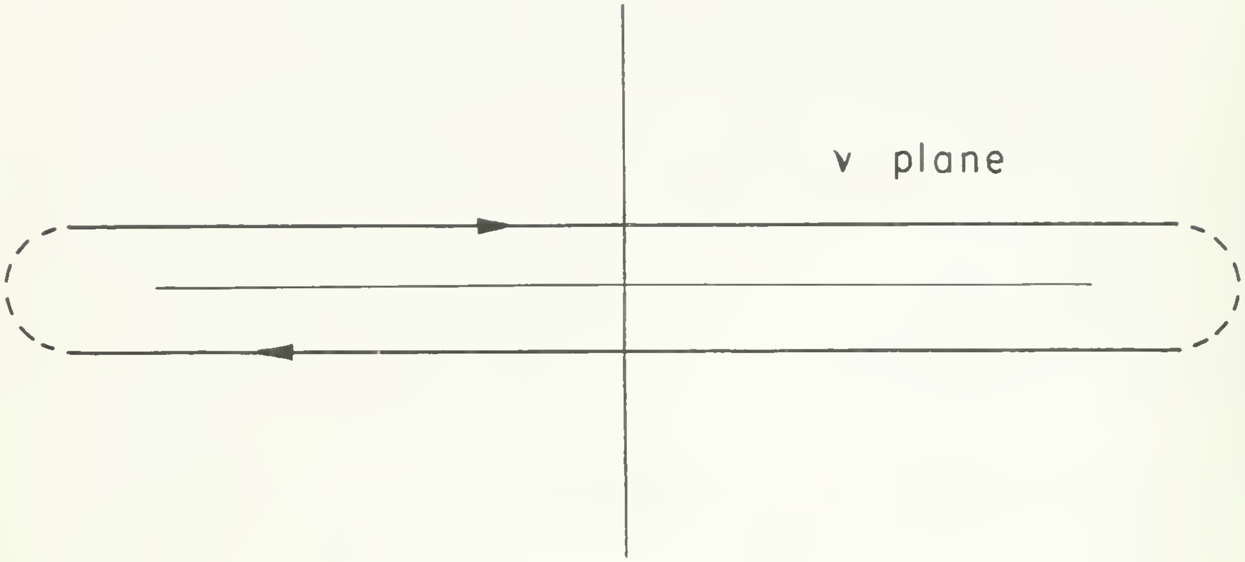


Figure 9: The path of integration for the Watson type integral representation of the field diffracted by a circular cylinder.

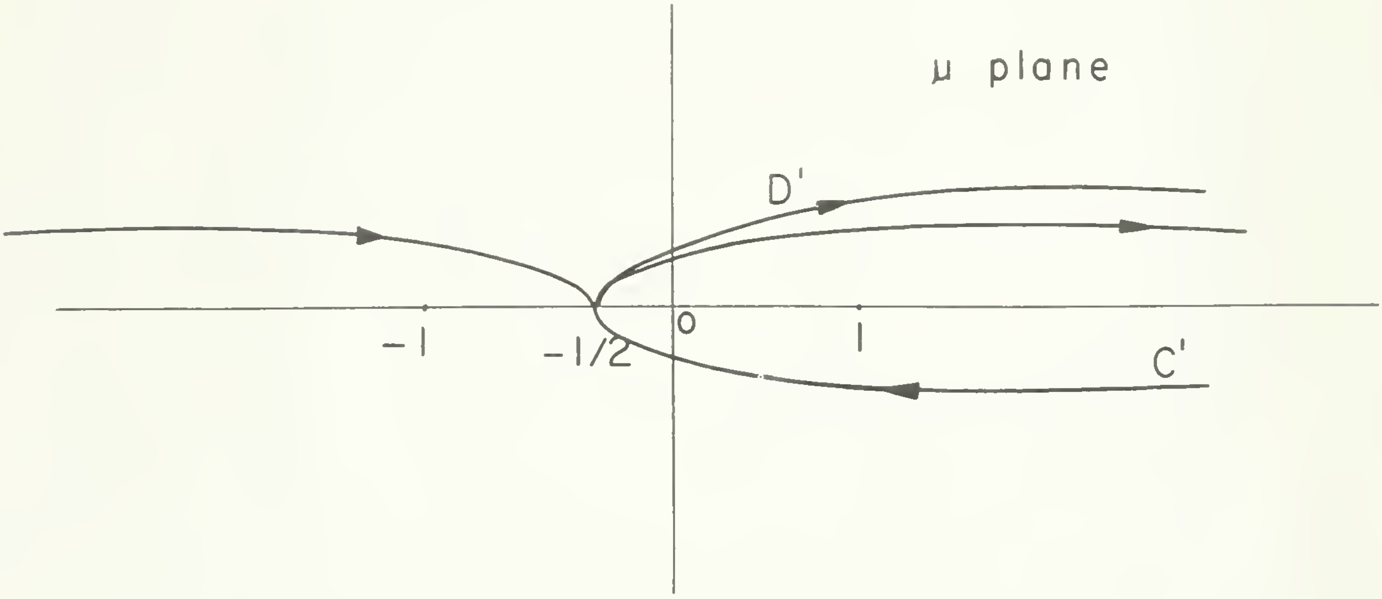


Figure 10: The paths of integration for the integral representation of the field diffracted by a sphere. The contour C' encircles the positive real axis, while D' coincides with the upper branch of C' for Re $\mu \geq -\frac{1}{2}$ and is as shown for Re $\mu \leq -\frac{1}{2}$.

(2) becomes*

$$(3)\qquad u = -\frac{1}{4}\int_{\Gamma} \frac{\cos\nu(\theta-\pi)}{\sin\nu\pi}\,\frac{H_\nu^{(1)}(kr_>)}{\Omega H_\nu^{(1)}(ka)}\left[\Omega H_\nu^{(1)}(ka)J_\nu(kr_<)-\Omega J_\nu(ka)H_\nu^{(1)}(kr_<)\right] d\nu.$$

Here Γ is a contour in the upper half plane running parallel to the real axis.

The contour Γ can be closed in the upper half-plane provided the point r, θ lies in the geometrical shadow. Then the integral can be evaluated by computing the residues at the zeroes ν_m of $\Omega H_\nu^{(1)}(ka)$. This yields

$$(4)\qquad u = \frac{i\pi}{4}\sum_m \frac{\cos\nu_m(\theta-\pi)}{\sin\nu_m\pi}\,\frac{\Omega H_{\nu_m}^{(2)}(ka)}{\frac{\partial}{\partial\nu}\Omega H_{\nu_m}^{(1)}(ka)}\;H_{\nu_m}^{(1)}(kr)\,H_{\nu_m}^{(1)}(k\rho).$$

When we asymptotically expand the Hankel functions appearing in (4), this equation takes the form

$$u \sim -\frac{i\exp\left[ik\left\{(r^2-a^2)^{1/2}+(\rho^2-a^2)^{1/2}\right\}\right]}{2k(r^2-a^2)^{1/4}(\rho^2-a^2)^{1/4}}\;\cdot$$

$$(5)\qquad \cdot\sum_m \frac{\exp\left[i\nu_m\theta\right]+\exp\left[i\nu_m(2\pi-\theta)\right]}{1-\exp\left[2\pi i\nu_m\right]}$$

$$\cdot\exp\left[-i\nu_m\left(\cos^{-1}(a/r)+\cos^{-1}(a/\rho)\right)\right]\;\frac{\Omega H_{\nu_m}^{(2)}(ka)}{\frac{\partial}{\partial\nu}\Omega H_{\nu_m}^{(1)}(ka)}\;.$$

Equation (5) contains the leading terms in the asymptotic expansion of the field in the shadow region. A slightly different evaluation of the integral in (3) leads to a similar result for the diffracted field in the lit region plus additional terms which represent the geometrical optics field u_g. This evaluation is carried out for the field diffracted by a sphere in Section 3. The result (5) has been compared with the geometrical result in Part I, Section 4.

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* Equation (3) is exact in Cases I, II, and III. In Case IV, however, where the operation Ω is a function of ν, other integrals also appear. In the case of opaque obstacles (e.g., lossy or highly conducting ones) those integrals are negligible compared with that retained in (3). Consequently we neglect those integrals in our further developments.

The values of ν_m and of $\Omega H_{\nu_m}^{(2)}(ka)\left[\frac{\partial}{\partial\nu}\Omega H_{\nu_m}^{(1)}(ka)\right]^{-1}$ appearing in (5) depend upon the boundary conditions. In each case ν_m has the form $\nu_m = ka+q_m(ka/6)^{1/3}e^{i\pi/3}$. The quantity q_m is the root of an equation given in Table I, page 14 . The function A(x) appearing in this table is the Airy function defined by

$$(6) \qquad A(x) = \int_0^\infty \cos(\tau^3 - x\tau)d\tau \ .$$

In Case IV the results in the table have been further simplified by assuming $|k_1| \gg |k|$. The factor $\Omega H_{\nu_m}^{(2)}(ka)\left[\frac{\partial}{\partial\nu}\Omega H_{\nu_m}^{(1)}(ka)\right]^{-1}$ is given in the last column of the table.

3. Scalar diffraction by a sphere

We will now treat the diffraction of a spherical wave by a sphere of radius a, using the same techniques as were used in the previous section. The formulation of the problem is the same as in that section, the delta function and the radiation condition being modified to correspond to the present three-dimensional case. The source is located on the polar axis at $r = \rho$, $\theta = 0$. The symmetry of the problem ensures that the solution is independent of the azimuth ϕ.

Upon expanding $u(r,\theta)$ in spherical wave functions and applying the boundary conditions, we obtain the solution

$$(7) \qquad u = \frac{ik}{4\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)h_n^{(1)}(kr_>)\left[j_n(kr_<)-\frac{\Omega j_n(ka)}{\Omega h_n^{(1)}(ka)}h_n^{(1)}(kr_<)\right] .$$

The functions $h_n^{(i)}(x)$ and $j_n(x)$ are the spherical Hankel and Bessel functions

$$h_n^{(i)}(x) = \sqrt{\frac{\pi}{2x}}\ H_{n+\frac{1}{2}}^{(i)}(x); \qquad j_n(x) = \sqrt{\frac{\pi}{2x}}\ J_{n+\frac{1}{2}}(x) .$$

The operator Ω is the same operator that appeared in the cylinder solution. However, in Case IV the Bessel functions in Ω are to be replaced by the spherical Bessel functions and the function $F(x)$ is a spherical cylinder function.

We again use the Watson transformation to transform (7) into a form which is numerically useful for large ka. Let C' be a contour encircling the positive real axis as shown in Figure 10. The sum in (7) can be written as

$$(8) \qquad u = -\frac{k}{8\pi}\int_{C'} \frac{(2\mu+1)P_\mu(-\cos\theta)h_\mu^{(1)}(kr_>)\left[j_\mu(kr_<)\,\Omega\, h_\mu^{(1)}(ka) - \Omega\, j_\mu(ka)h_\mu^{(1)}(kr_<)\right]}{\sin\mu\pi \qquad h_\mu^{(1)}(ka)}\,d\mu\,.$$

Upon replacing μ by $-\mu-1$ on that part of the contour C' for which $\operatorname{Im}\mu < 0$ we can rewrite the above integral as an integral over the contour D' shown in Figure 10[*]. The integral in (8) can be evaluated in terms of its residues at the zeros μ_m of $\Omega\, h_{\mu_m}^{(1)}(ka)$ provided the point (r,θ) lies in the shadow. Then the integral becomes

$$(9) \qquad u = \frac{ik}{8}\sum_m \frac{(2\mu_m+1)P_{\mu_m}(-\cos\Theta)}{\sin\mu_m\pi}\; h_{\mu_m}^{(1)}(kr)h_{\mu_m}^{(1)}(k\rho)\; \frac{\Omega\, h_{\mu_m}^{(2)}(ka)}{\frac{\partial}{\partial\mu}\,\Omega\, h_{\mu_m}^{(1)}(ka)}\,.$$

If we set $\nu_m = \mu_m + \frac{1}{2}$ then the ν_m entering (9) are asymptotically (for ka large) equal to the ν_m occurring in the cylinder solution. We also have

$$\frac{\Omega\, h_{\mu_m}^{(2)}(ka)}{\frac{\partial}{\partial\mu}\,\Omega h_{\mu_m}^{(1)}(ka)} \sim \frac{\Omega H_{\nu_m}^{(2)}(ka)}{\frac{\partial}{\partial\nu}\cdot\Omega H_{\nu_m}^{(1)}(ka)}\,.$$

Therefore, upon asymptotically expanding the Hankel and Legendre functions appearing in (9), we obtain

---

[*] Again this is exact in Cases I, II, and III and represents an approximation based upon the opacity of the obstacle for the other boundary condition.

$$u \sim - \frac{a^{1/2}\exp\left[ik\left\{(r^2-a^2)^{1/2}+(\rho^2-a^2)^{1/2}\right\}\right]}{(8\pi kr\rho\sin\theta)^{1/2}(r^2-a^2)^{1/4}(\rho^2-a^2)^{1/4}}$$

$$(10) \qquad \sum_m \frac{\exp\left[i\nu_m(2\pi-\theta)-\frac{i\pi}{4}\right]+\exp\left[i\nu_m\theta+\frac{i\pi}{4}\right]}{1+\exp\left[2\pi i\nu_m\right]} \cdot$$

$$\cdot \exp\left[-i\nu_m\left[\cos^{-1}(a/r)+\cos^{-1}(a/\rho)\right]\right] \frac{\Omega H^{(2)}_{\nu_m}(ka)}{\frac{\partial}{\partial\nu}\Omega H^{(1)}_{\nu_m}(ka)} .$$

The values of $\Omega H^{(2)}_{\nu_m}(ka)\left[\frac{\partial}{\partial\nu}\Omega H_{\nu_m}(ka)\right]^{-1}$ and equations for q_m, from which $\nu_m = ka + q_m(ka/6)^{1/3}\exp(i\pi/3)$ can be found, are given in Table I, page 14.

The above result for the field in the shadow does not apply on the caustics $r = a$ and $\theta = \pi$ because the asymptotic expansions used for the Hankel and Legendre functions do not apply there.

We will now apply a method developed by W. Franz [10] to compute the field in the lit region by means of the Watson transformation. To carry out this evaluation we write the Legendre function $P_\mu(-\cos\theta)$, which appears in the integral over the contour D', in the form

$$P_\mu(-\cos\theta) = e^{i\pi\mu} P_\mu(\cos\theta) - 2i \sin\mu\pi \, Q^{(2)}_\mu(\theta).$$

We then split the integral into the sum of two integrals, one containing $e^{i\pi\mu}P_\mu(\cos\theta)$ and the other containing $-2i\sin\mu\pi\, Q^{(2)}_\mu(\theta)$. The first integral can be evaluated by residues to yield the diffracted field. The second integral can be evaluated by separating out the incident field and then using a saddle point approximation to yield the reflected field. Upon applying this procedure, the field in the illuminated region is found to be

$$
\begin{aligned}
u \sim u_g + {} & \frac{i a^{1/2} \exp\left[ik\left\{(r^2-a^2)^{1/2} + (\rho^2-a^2)^{1/2}\right\}\right]}{(8\pi k r \rho \sin\theta)^{1/2}(r^2-a^2)^{1/4}(\rho^2-a^2)^{1/4}} \\
(11) \qquad & \sum_m \frac{\exp\left[i\nu_m(\theta+2\pi) - \frac{i\pi}{4}\right] + \exp\left[i\nu_m(2\pi-\theta) + \frac{i\pi}{4}\right]}{1 + \exp\left[2i\pi\nu_m\right]} \\
& \cdot \exp\left[-i\nu_m\left[\cos^{-1}(a/r) + \cos^{-1}(a/\rho)\right]\right] \frac{\Omega H^{(2)}_{\nu_m}(ka)}{\frac{\partial}{\partial\nu}\Omega H^{(1)}_{\nu_m}(ka)} \quad .
\end{aligned}
$$

This result is not valid on the caustics $r = a$ and $\theta = 0$ for the reason given above. The diffracted fields in (10) and (11) have been compared with the geometrical results in Part I, Section 6.

4. Electromagnetic diffraction by a finitely conducting circular cylinder at oblique incidence

Consider a circular cylinder having propagation constant k_1 located in a medium of propagation constant k. Suppose a plane electromagnetic wave of angular frequency ω is obliquely incident upon the cylinder. Let us introduce a rectangular coordinate system in such a way that the generators of the cylinder lie along the z axis and the direction of propagation of the wave is parallel to the (y,z) plane. If α is the angle between the direction of incidence and the z axis then the direction of propagation $\overrightarrow{d_1^o}$ is given by $\overrightarrow{d_1^o} = (0, \sin\alpha, \cos\alpha)$. Now, if the wave is linearly polarized and has an electric vector making an angle $\frac{\pi}{2} - \theta$ with the (y,z) plane, it can be written as

$$E_x = A \sin\theta \, e^{ik(y \sin\alpha + z \cos\alpha)}$$

$$E_y = A \cos\theta \cos\alpha \, e^{ik(y \sin\alpha + z \cos\alpha)}$$

$$E_z = -A \cos\theta \sin\alpha \, e^{ik(y \sin\alpha + z \cos\alpha)}$$

and

$$H_x = -\frac{Ak}{\mu\omega} \cos\theta\, e^{ik(y \sin\alpha + z\cos\alpha)}$$

$$H_y = \frac{Ak}{\mu\omega} \sin\theta \cos\alpha\, e^{ik(y \sin\alpha + z\cos\alpha)}$$

$$H_z = -\frac{Ak}{\mu\omega} \sin\theta \sin\alpha\, e^{ik(y \sin\alpha + z\cos\alpha)}.$$

In order to simplify the subsequent analysis we break up the field into two partial fields E^1, H^1 and E^2, H^2 defined by

$$E_y^{(1)} = E_y \qquad H_x^{(1)} = H_x$$

$$E_z^{(1)} = E_z \qquad E_x^{(1)} = H_y^{(1)} = H_z^{(1)} = 0$$

and

$$E_x^{(2)} = E_x \qquad H_z^{(2)} = H_z$$

$$H_y^{(2)} = H_y \qquad E_y^{(2)} = E_z^{(2)} = H_x^{(2)} = 0.$$

Because of the symmetry properties of Maxwell's equations it is sufficient to consider only the problem for which the incident field is given by E^1, H^1. This is due to the fact that if we interchange E and H and replace μ by $-k^2/\omega^2\mu$ Maxwell's equations are left invariant*. Therefore, to obtain the field corresponding to the incident field E^2, H^2 from the field corresponding to the incident field E^1, H^1 all we need do is interchange E and H, replace μ by $-k^2/\omega^2\mu$ and then replace A by $\frac{A\sin\theta}{\cos\theta}\ \frac{k}{\mu\omega}$.

---

* In the exterior medium we take Maxwell's equations in the form $\nabla \times E = i\mu\omega H$ and $\nabla \times H = \frac{-ik^2}{\mu\omega} E$. In the interior of the cylinder μ should be replaced by μ_1 and k by k_1.

To calculate the diffracted field due to the incident field E^1, H^1 we first introduce polar coordinates (ρ,ϕ) in the (x,y) plane. Next we define two potential functions, $u(\rho,\phi)$ and $v(\rho,\phi)$, both satisfying the equations

$$(\nabla^2 + k'^2)\binom{u}{v} = 0 \qquad \rho \geq a$$

$$(\nabla^2 + k_1'^2)\binom{u}{v} = 0 \qquad 0 \leq \rho \leq a.$$

Here $k' = k \sin\alpha$ and $k_1' = (k_1^2 - k^2\cos^2\alpha)^{1/2}$. In terms of these potentials we define the vectors $E'(\rho,\phi)$ and $H'(\rho,\phi)$ in the region $\rho \geq a$ by

$$(12)\quad \begin{cases} E_r' = ik\cos\alpha \dfrac{\partial u}{\partial r} + \dfrac{i\mu\omega}{r} \dfrac{\partial v}{\partial \phi} \\ E_\phi' = \dfrac{ik\cos\alpha}{r} \dfrac{\partial u}{\partial \phi} - i\mu\omega \dfrac{\partial v}{\partial r} \\ E_z' = k'^2 u \end{cases}$$

$$(13)\quad \begin{cases} H_r' = -\dfrac{ik^2}{\mu\omega r} \dfrac{\partial u}{\partial \phi} + ik\cos\alpha \dfrac{\partial v}{\partial r} \\ H_\phi' = \dfrac{ik^2}{\mu\omega} \dfrac{\partial u}{\partial r} + \dfrac{ik\cos\alpha}{r} \dfrac{\partial v}{\partial \phi} \\ H_z' = k'^2 v\ . \end{cases}$$

Then the vectors $E = E'(\rho,\phi)e^{ikz\cos\alpha}$ and $H = H'(\rho,\phi)e^{ikz\cos\alpha}$ are solutions of Maxwell's equations[11].

As an example of this representation, the incident field E^1, H^1 can be represented in terms of the potentials u_i and v_i defined by

$$(14)\qquad u_i(\rho,\phi) = -(k')^{-2}A\cos\theta\ \sin\alpha\ e^{ik'y}$$

$$(15)\qquad v_i(\rho,\phi) = 0\ .$$

In the interior medium the expressions for E_r' and E_ϕ' are obtained by replacing μ by μ_1 in the above equations. The values of E_z' and H_z' are obtained by replacing k' by k_1'. The expressions for H_r' and H_ϕ' are obtained by replacing μ by μ_1 and k by k_1 in the terms multiplying the derivatives of u and leaving the terms multiplying the derivatives of v unchanged.

When E and H are represented in the above manner, the continuity of the tangential components of E and H across the cylinder imposes conditions on the potentials u and v. These conditions are

$$k'^2 u\Big]_{\rho=a+} = k_1'^2 u\Big]_{\rho=a-}$$

$$k'^2 v\Big]_{\rho=a+} = k_1'^2 v\Big]_{\rho=a-}$$

$$\left[\frac{ik\cos\alpha}{a}\frac{\partial u}{\partial \phi} - i\mu\omega\frac{\partial v}{\partial r}\right]_{\rho=a+} = \left[\frac{ik\cos\alpha}{a}\frac{\partial u}{\partial \phi} - i\mu_1\omega\frac{\partial v}{\partial r}\right]_{\rho=a-}$$

$$\left[\frac{ik^2}{\mu\omega}\frac{\partial u}{\partial r} + \frac{ik\cos\alpha}{a}\frac{\partial v}{\partial \phi}\right]_{\rho=a+} = \left[\frac{ik_1^2}{\mu_1\omega}\frac{\partial u}{\partial r} + \frac{ik\cos\alpha}{a}\frac{\partial v}{\partial \phi}\right]_{\rho=a-}$$

We now write $u = u_i + u_s$ and $v = v_i + v_s$ where both u_s and v_s satisfy the radiation condition. Next we expand the field in cylindrical wave functions. Upon taking account of the form of u_i and v_i and the radiation condition we obtain for this expansion

$$u = -\frac{A\cos\theta\sin\alpha}{k'^2}\sum_{n=-\infty}^{\infty}\left[J_n(k'\rho) - a_n H_n^{(1)}(k'\rho)\right]e^{in\phi} \qquad \rho \geq a$$

(16)

$$= -\frac{A\cos\theta\sin\alpha}{k'^2}\sum_{n=-\infty}^{\infty} b_n J_n(k_1'\rho)\, e^{in\phi} \qquad 0 \leq \rho \leq a$$

(17)
$$v = -\frac{A\cos\theta\sin\alpha}{k'^2}\sum_{n=-\infty}^{\infty} c_n H_n^{(1)}(k'\rho)\, e^{in\phi} \qquad \rho \geq a$$

$$= -\frac{A\cos\theta\sin\alpha}{k'^2}\sum_{n=-\infty}^{\infty} d_n J_n(k_1'\rho)\, e^{in\phi} \qquad 0 \leq \rho \leq a \quad .$$

The constants a_n, b_n, c_n, and d_n are to be obtained by applying the four boundary conditions. Upon doing this we find that the constants a_n and c_n have the values

(18)
$$a_n = \frac{J_n(k'a)}{\triangle H_n^{(1)}(k'a)}\left\{\left(\frac{\mu_1}{\mu k_1'a}\,\frac{J_n'(k_1'a)}{J_n(k_1'a)} - \frac{1}{k'a}\,\frac{H_n^{(1)\prime}(k'a)}{H_n^{(1)}(k'a)}\right)\left(\frac{k_1^2}{k^2}\,\frac{\mu}{\mu_1 k_1'a}\,\frac{J_n'(k_1'a)}{J_n(k_1'a)} - \frac{1}{k'a}\,\frac{J_n'(k'a)}{J_n(k'a)}\right) - \left(\frac{1}{(k'a)^2} - \frac{1}{(k_1'a)^2}\right)^2 n^2\cos^2\alpha\right\} ,$$

(19)
$$c_n = \frac{2\,k\,n\cos\alpha}{\triangle\,\mu\omega\pi(k'a)^2}\,\frac{1}{\left[H_n^{(1)}(k'a)\right]^2}\left(\frac{1}{(k'a)^2} - \frac{1}{(k_1'a)^2}\right) ,$$

where

$$\triangle = \left(\frac{\mu_1}{\mu k_1'a}\,\frac{J_n'(k_1'a)}{J_n(k_1'a)} - \frac{1}{k'a}\,\frac{H_n^{(1)\prime}(k'a)}{H_n^{(1)}(k'a)}\right)\left(\frac{k_1^2}{k^2}\,\frac{\mu}{\mu_1 k_1'a}\,\frac{J_n'(k_1'a)}{J_n(k_1'a)} - \frac{1}{k'a}\,\frac{H_n^{(1)\prime}(k'a)}{H_n^{(1)}(k'a)}\right)$$

$$-\left(\frac{1}{(k'a)^2} - \frac{1}{(k_1'a)^2}\right)^2 n^2\cos^2\alpha \; .$$

These results agree with those given by J.R. Wait[13] who obtained them without making use of potentials.

Let us now expand these results asymptotically for large values of ka. First we observe that the term

$$\left[\frac{1}{(k'a)^2} - \frac{1}{(k_1'a)^2}\right] n$$

tends to zero as ka and k_1a become infinite*. Therefore $c_n \sim 0$ and

$$a_n \sim \frac{J_n(k'a)}{H_n^{(1)}(k'a)} \left(\frac{k_1^2}{k^2} \; \frac{\mu}{\mu_1 k_1'} \; \frac{J_n'(k_1'a)}{J_n(k_1'a)} - \frac{1}{k'} \; \frac{J_n'(k'a)}{J_n(k'a)} \right)$$

$$\left(\frac{k_1^2}{k^2} \; \frac{\mu}{\mu_1 k_1'} \; \frac{J_n'(k_1'a)}{J_n(k_1'a)} - \frac{1}{k'} \; \frac{H_n^{(1)\prime}(k'a)}{H_n^{(1)}(k'a)} \right)^{-1}. \tag{20}$$

When the above value of a_n is inserted into (16) the resulting function $u(\rho,\emptyset)$ can be shown to be the solution of the following scalar problem

$$(\nabla^2 + k'^2)u = 0 \qquad \rho \geq a$$

$$(\nabla^2 + k_1'^2)u = 0 \qquad 0 \leq \rho \leq a$$

$$u = u_i + u_s$$

$$\lim_{\rho \to \infty} \rho^{1/2} \left(\frac{\partial u_s}{\partial \rho} - ik' u_s \right) = 0$$

$$\frac{\partial u}{\partial \rho} \text{ continuous at } \rho = a$$

$$bu\Big]_{\rho = a+} = b_1 u\Big]_{\rho = a-}$$

Here b and b_1 are given by

$$b = k'^2 \mu / k^2 \tag{21}$$

$$b_1 = k_1'^2 \mu_1 / k_1^2 . \tag{22}$$

---

[*] If this term is retained it can be shown that it will not influence the leading term of the asymptotic expansion. Consequently, we set it equal to zero at this point to facilitate the subsequent analysis.

It is to be noted that in the special case in which k_1 becomes infinite, which occurs when the conductivity of the cylinder becomes infinite, (20) simplifies to

$$a_n = \frac{J_n(k'a)}{H_n^{(1)}(k'a)} \quad .$$

In this case the boundary condition on $u(\rho,\phi)$ becomes $u(a,\phi) = 0$.

We now proceed in exactly the same way as we did in Section 2. For a point ρ, ϕ in the shadow, we find that for large values of ka and k_1a, provided $|k_1| >> |k|$, u is asymptotically given by

$$(23) \qquad u \sim -(k')^{-2} A \cos\theta \sin\alpha \left[w(\rho,\phi) + w(\rho,\pi-\phi)\right].$$

Here w is defined by

$$(24) \qquad w(\rho,\phi) = \sum_m \frac{D_m^2}{(\rho^2-a^2)^{1/2}} \; \frac{\exp\left[ik'(\rho^2-a^2)^{1/2} + i\nu_m(\phi - \cos^{-1}(\frac{a}{\rho}))\right]}{1 - \exp\left[2\pi i\nu_m\right]} \quad .$$

The values of D_m and ν_m can be determined from Table I, Case IV with k replaced by k', k_1 replaced by k_1' and b and b_1 given by (21) and (22). A similar result can be obtained for points in the lit region.

We now calculate the vector E' by substituting (23) into (12) and keeping only the highest order terms in k. Upon using the fact that $\nu_m \sim k'a$, we obtain

$$E_r' \sim A(\rho^2-a^2)^{1/2}\rho^{-1}\cos\theta \cos\alpha\left[w(\rho,\phi) + w(\rho,\pi-\phi)\right]$$

$$(25) \qquad E_\phi' \sim Aa\rho^{-1} \cos\theta \cos\alpha\left[w(\rho,\phi) - w(\rho,\pi-\phi)\right]$$

$$E_z' \sim -A \cos\theta \sin\alpha\left[w(\rho,\phi) + w(\rho,\pi-\phi)\right] .$$

The electric field E in the shadow is obtained by multiplying (25) by $e^{ikz\cos\alpha}$.

The corresponding result for the incident field E^2, H^2 can be obtained from this result by performing in (25) the substitutions described before. The result is

$$E'_r \sim A\, a\, \rho^{-1} \sin\theta \left[v(\rho,\phi) - v(\rho,\pi-\phi) \right]$$

(26) $$E'_\phi \sim -A(\rho^2-a^2)^{1/2}\, \rho^{-1} \sin\theta \left[v(\rho,\phi) + v(\rho,\pi-\phi) \right]$$

$$E'_z \sim 0 \ .$$

The function $v(\rho,\phi)$ in (26) is given by (24) with D_m and ν_m determined from Table I, Case IV with b and b_1 given by

(27) $$b = k'^2/\mu \qquad\qquad b_1 = k_1'^2/\mu_1 \ .$$

These values of b and b_1 are obtained by performing the required substitutions for μ and μ_1 in (21) and (22). Then the common factor $-\omega^2$ is cancelled from both of the resulting equations since only the ratio b/b_1 occurs in the problem. The electric field E in the shadow is now given by multiplying (26) by $e^{ikz\cos\alpha}$.

The total field in the shadow is, apart from the factor $e^{ikz\cos\alpha}$, the sum of the fields (25) and (26) due to the incident fields E^1, H^1 and E^2, H^2. The field in the illuminated region consists of a geometrical optics field and a diffracted field of the same form as that found above. The above result for the diffracted field has been compared with the geometrical result in Part I, Section 9.

5. Electromagnetic diffraction by a sphere

We will now consider the problem of diffraction of a linearly polarized plane electromagnetic wave by a sphere of radius a. By using techniques similar to those used in the previous sections we will solve for the field and evaluate it asymptotically for large values of ka.

First we introduce a coordinate system with the origin at the center of the sphere. We orient the axes in such a way that the direction of propagation of the incident wave is the positive z axis and the direction of the incident E vector is the x axis. The incident electromagnetic field can then be written as

$E_y = E_z = H_x = H_y = 0$ and

$$E_x = A\, e^{ikz} \tag{28}$$

$$H_y = \frac{Ak}{\mu\omega}\, e^{ikz} \,. \tag{29}$$

Upon introducing a spherical coordinate system we find that the components of the diffracted field are given by [12]

$$E_r = \frac{iA\cos\phi}{kr} \frac{\partial}{\partial\theta} \sum_{n=1}^{\infty} i^n (2n+1) P_n(\cos\theta) \left(j_n(kr) - b_n h_n^{(1)}(kr) \right) \tag{30}$$

$$\begin{aligned} E_\theta = {} & \frac{iA\cos\phi}{kr} \frac{\partial^3}{\partial r \partial^2 \theta} \left[r \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} P_n(\cos\theta) \left(j_n(kr) - b_n h_n^{(1)}(kr) \right) \right] \\ & - \frac{A\cos\phi}{\sin\theta} \frac{\partial}{\partial\theta} \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} P_n(\cos\theta) \left(j_n(kr) - a_n h_n^{(1)}(kr) \right) \end{aligned} \tag{31}$$

$$\begin{aligned} E_\phi = {} & - \frac{iA\sin\phi}{\sin\theta\, kr} \frac{\partial^2}{\partial r \partial\theta} \left[r \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} P_n(\cos\theta) \left(j_n(kr) - b_n h_n^{(1)}(kr) \right) \right] \\ & + A\sin\phi \frac{\partial^2}{\partial\theta^2} \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} P_n(\cos\theta) \left(j_n(kr) - a_n h_n^{(1)}(kr) \right) . \end{aligned} \tag{32}$$

Here a_n and b_n are defined by

$$a_n = \frac{\mu j_n(ka) \frac{d}{dx}\Big[x j_n(x) \Big]_{x=k_1 a} - \mu_1 j_n(k_1 a) \frac{d}{dx}\Big[x j_n(x) \Big]_{x=ka}}{\mu h_n^1(ka) \frac{d}{dx}\Big[x j_n(x) \Big]_{x=k_1 a} - \mu_1 j_n(k_1 a) \frac{d}{dx}\Big[x h_n^1(x) \Big]_{x=ka}} \tag{33}$$

(34) $$b_n = \frac{\mu_1 j_n(ka)\,\frac{d}{dx}\Big[xj_n(x)\Big]_{x=k_1a} - \mu(k_1/k)^2\;\; j_n(k_1a)\,\frac{d}{dx}\Big[xj_n(x)\Big]_{x=ka}}{\mu_1 h_n^{(1)}(ka)\frac{d}{dx}\Big[xj_n(x)\Big]_{x=k_1a} - \mu(k_1/k)^2\;\; j_n(k_1a)\frac{d}{dx}\Big[xh_n^{(1)}(x)\Big]_{x=ka}} .$$

For large values of x, $\frac{d}{dx}\Big[xj_n(x)\Big]$ is asymptotic to $xj_n^{'}(x)$ and $\frac{d}{dx}\Big[xh_n^1(x)\Big]$ is asymptotic to $xh_n^{(1)'}(x)$. When these results are used in (33) and (34) for ka and k_1a large, these equations become

(35) $$a_n \quad \frac{\mu k_1 j_n(ka)j_n^{'}(k_1a) - \mu_1 k j_n(k_1a)j_n^{'}(ka)}{\mu k_1 h_n^{(1)}(ka)j_n^{'}(k_1a) - \mu_1 k j_n(k_1a)h_n^{(1)'}(ka)}$$

(36) $$b_n = \frac{\frac{k^2}{\mu}\, k_1 j_n(ka)j_n^{'}(k_1a) - \frac{k_1^2}{\mu_1}\, k j_n(k_1a)j_n^{'}(ka)}{\frac{k^2}{\mu}\, k_1 h_n^{(1)}(ka)j_n^{'}(k_1a) - \frac{k_1^2}{\mu_1}\, k j_n(k_1a)h_n^{(1)'}(ka)} .$$

By the same technique as was used in the previous section, it can be shown that for ka and k_1a large and the point r, θ in the shadow, the following asymptotic relations hold

(37) $$\sum_{n=1}^{\infty} i^n(2n+1)P_n(\cos\theta)\Big(j_n(kr) - b_n h_n^{(1)}(kr)\Big) \sim v(r,\theta) + e^{-i\frac{\pi}{2}}\, v(r,-\theta)$$

(38) $$\sum_{n=1}^{\infty} i^n\, \frac{(2n+1)}{n(n+1)}\, P_n(\cos\theta)\Big(j_n(kr) - b_n h_n^{(1)}(kr)\Big) \sim (ka)^{-2}\Big[v(r,\theta) + e^{-i\frac{\pi}{2}}\, v(r,-\theta)\Big]$$

(39) $$\sum_{n=1}^{\infty} i^n\, \frac{(2n+1)}{n(n+1)}\, P_n(\cos\theta)\Big(j_n(kr) - a_n h_n^{(1)}(kr)\Big) \sim (ka)^{-2}\Big[u(r,\theta) + e^{-i\frac{\pi}{2}}\, u(r,-\theta)\Big] .$$

Here $u(r,\theta)$ and $v(r,\theta)$ are defined by

$$(40)\quad \begin{matrix} u(r,\theta) \\ v(r,\theta) \end{matrix} = a^{\frac{1}{2}} \left[r(r^2-a^2)^{1/2}|\sin\theta|\right]^{-1/2} \exp\left[ik(r^2-a^2)^{1/2}\right] \sum_m \frac{D_m^2 \exp\left[i\nu_m(\frac{\pi}{2} - \theta - \cos^{-1}(a/r)\right]}{1 + \exp\left[i\nu_m \pi\right]} .$$

The values of ν_m and D_m in the expression for $u(r,\theta)$ are to be determined by the entries in Table I for Case IV with $b = \mu$ and $b_1 = \mu_1$. The values of ν_m and D_m in the expression for $v(r,\theta)$ are to be determined from the same table by setting $b = k^2/\mu$ and $b_1 = k_1^2/\mu_1$. These facts can be seen by comparing (35) and (36) with the expressions for the Ω operator in Case IV given on page 14 .

Now to calculate the electromagnetic field we insert (40) into (37)-(39) and use the results in (30)-(32). The resulting expressions for the electric field components in the shadow are

$$(41)\qquad E_r \sim Aar^{-1} \cos \phi \left[v(r,\theta) - e^{-i\frac{\pi}{2}} v(r,-\theta)\right]$$

$$(42)\qquad E_\theta \sim A(r^2-a^2)^{1/2}\, r^{-1} \cos \phi \left[v(r,\theta) + e^{-i\frac{\pi}{2}} v(r,-\theta)\right]$$

$$(43)\qquad E_\phi \sim -A \sin \phi \left[u(r,\theta) + e^{-i\frac{\pi}{2}} u(r,-\theta)\right] .$$

In these expressions we have dropped lower order terms in ka. The corresponding results for the field in the illuminated region are obtained as in Section 3. The above results are not valid on the caustics or on the shadow boundary. These results have been compared with the geometrical results in Part I, Section 8.

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